COMPSCI 514: Algorithms for Data Science

Cameron Musco
University of Massachusetts Amherst. Fall 2023.
Lecture 5

Logistics

- Problem Set 1 is due this Friday at 11:59pm.
- A useful technique: to prove that $|a| \leq b$, prove both sides: that $a \leq b$ and that $a \geq -b$.
- Quiz question on class pacing:
    - Way too fast: 9.
    - A bit too fast: 43.
    - Just right: 64.
    - A bit too slow: 3.
    - Way too slow: 0.

Last Time

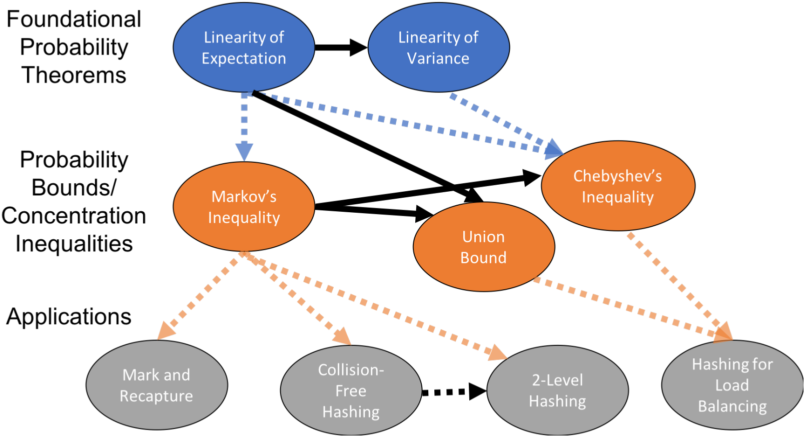
Last Class:

- 2-universal and pairwise independent hash functions.
- The union bound.
- Application to hashing for load balancing.

This Time:

- Exponential concentration bounds and the central limit theorem.

Concept Map



Foundational Probability Theorems
Linearity of Expectation
Linearity of Variance
Probability Bounds/ Concentration Inequalities
Markov's Inequality
Union Bound
Chebyshev's Inequality
Applications
Mark and Recapture
Collision-Free Hashing
2-Level Hashing
Hashing for Load Balancing

Quiz Questions

My (not very popular) photo hosting service receives 5 download requests per day. Each download request is completed successfully with probability 0.98. Give an upper bound on the probability that my service fails to complete at least one request successfully. Hint: do not assume independence of the request completions.

Answer:

Check

If the failures were independent: $1 - .98^5 = 0.096$. Only a bit smaller than the upper bound of 0.1.

More Union Bound Intuition

Flipping Coins

We flip $n = 100$ independent coins, each are heads with probability 1/2 and tails with probability 1/2. Let H be the number of heads.

$$\mathbb{E}[\mathsf{H}] = \frac{n}{2} = 50 \text{ and } \mathrm{Var}[\mathsf{H}] = \frac{n}{4} = 25$$

Markov's:	Chebyshev's:	In Reality:
$\Pr(\mathsf{H} \geq 60) \leq .833$	$\Pr(\mathsf{H} \geq 60) \leq .25$	$\Pr(\mathsf{H} \geq 60) = 0.0284$
$\Pr(\mathsf{H} \geq 70) \leq .714$	$\Pr(\mathsf{H} \geq 70) \leq .0625$	$\Pr(\mathsf{H} \geq 70) = .000039$
$\Pr(\mathsf{H} \geq 80) \leq .625$	$\Pr(\mathsf{H} \geq 80) \leq .0278$	$\Pr(\mathsf{H} \geq 80) < 10^{-9}$

H has a simple Binomial distribution, so can compute these probabilities exactly.

Tighter Concentration Bounds

To be fair.... Markov and Chebyshev's inequalities apply much more generally than to Binomial random variables like coin flips.

Can we obtain tighter concentration bounds that still apply to very general distributions?

- Markov's: $\Pr(X \geq t) \leq \frac{\mathbb{E}[X]}{t}$. First Moment.
- Chebyshev's: $\Pr(|X - \mathbb{E}[X]| \geq t) = \Pr(|X - \mathbb{E}[X]|^2 \geq t^2) \leq \frac{\text{Var}[X]}{t^2}$. Second Moment.
- What if we just apply Markov's inequality to even higher moments?

A Fourth Moment Bound

Consider any random variable X:

$$\Pr(|X - \mathbb{E}[X]| \geq t) = \Pr\left((X - \mathbb{E}[X])^4 \geq t^4\right) \leq \frac{\mathbb{E}\left[(X - \mathbb{E}[X])^4\right]}{t^4}.$$

Application to Coin Flips: Recall: $n = 100$ independent fair coins, H is the number of heads.

- Bound the fourth moment:

$$\mathbb{E}\left[(H - \mathbb{E}[H])^4\right] = \mathbb{E}\left[\left(\sum_{i=1}^{100} H_i - 50\right)^4\right] = \sum_{i,j,k,\ell} c_{ijk\ell}\mathbb{E}[H_i H_j H_k H_\ell] = 1862.5$$

  where $H_i = 1$ if coin flip i is heads and 0 otherwise. Then apply some messy calculations...
- Apply Fourth Moment Bound: $\Pr(|H - \mathbb{E}[H]| \geq t) \leq \frac{1862.5}{t^4}$.

Tighter Bounds

Chebyshev's:	4^{th} Moment:	In Reality:
$\Pr(H \geq 60) \leq .25$	$\Pr(H \geq 60) \leq .186$	$\Pr(H \geq 60) = 0.0284$
$\Pr(H \geq 70) \leq .0625$	$\Pr(H \geq 70) \leq .0116$	$\Pr(H \geq 70) = .000039$
$\Pr(H \geq 80) \leq .04$	$\Pr(H \geq 80) \leq .0023$	$\Pr(H \geq 80) < 10^{-9}$

Can we just keep applying Markov's inequality to higher and higher moments and getting tighter bounds?

- Yes! To a point.
- In fact – don't need to just apply Markov's to $|X - \mathbb{E}[X]|^k$ for some k. Can apply to any monotonic function $f(|X - \mathbb{E}[X]|)$.
- Why monotonic? $\Pr(|X - \mathbb{E}[X]| > t) = \Pr(f(|X - \mathbb{E}[X]|) > f(t))$.

H: total number heads in 100 random coin flips. $\mathbb{E}[H] = 50$.

Exponential Concentration Bounds

Moment Generating Function: Consider for any $t > 0$:

$$M_t(X) = e^{t\cdot(X-\mathbb{E}[X])} = \sum_{k=0}^{\infty} \frac{t^k(X-\mathbb{E}[X])^k}{k!}$$

- $M_t(X)$ is monotonic for any $t > 0$.
- Weighted sum of all moments, with t controlling how slowly the weights fall off (larger t = slower falloff).
- Choosing t appropriately lets one prove a number of very powerful exponential concentration bounds (exponential tail bounds).
- Chernoff bound, Bernstein inequalities, Hoeffding's inequality, Azuma's inequality, Berry-Esseen theorem, etc.
- We will not cover the proofs in this class (although we may in the next problem set).

Bernstein Inequality

Bernstein Inequality: Consider independent random variables $X_1, \ldots, X_n$ all falling in $[-M, M]$[-1,1]. Let $\mu = \mathbb{E}[\sum_{i=1}^n X_i]$ and $\sigma^2 = \text{Var}[\sum_{i=1}^n X_i] = \sum_{i=1}^n \text{Var}[X_i]$. For any $t \geq 0 s \geq 0$:

$$\Pr\left(\left|\sum_{i=1}^n X_i - \mu\right| \geq t\right) \leq 2\exp\left(-\frac{t^2}{2\sigma^2 + \frac{4}{3}Mt}\right).$$

$$\Pr\left(\left|\sum_{i=1}^n X_i - \mu\right| \geq s\sigma\right) \leq 2\exp\left(-\frac{s^2}{4}\right).$$

Assume that $M = 1$ and plug in $t = s \cdot \sigma$ for $s \leq \sigma$.

Compare to Chebyshev's: $\Pr\left(\left|\sum_{i=1}^n X_i - \mu\right| \geq s\sigma\right) \leq \frac{1}{s^2}$.

- An exponentially stronger dependence on s!

Comparision to Chebyshev's

Consider again bounding the number of heads H in $n = 100$ independent coin flips.

Chebyshev's:	Bernstein:	In Reality:
$\Pr(\mathrm{H} \geq 60) \leq .25$	$\Pr(\mathrm{H} \geq 60) \leq .21$	$\Pr(\mathrm{H} \geq 60) = 0.0284$
$\Pr(\mathrm{H} \geq 70) \leq .0625$	$\Pr(\mathrm{H} \geq 70) \leq .005$	$\Pr(\mathrm{H} \geq 70) = .000039$
$\Pr(\mathrm{H} \geq 80) \leq .04$	$\Pr(\mathrm{H} \geq 80) \leq 4^{-5}$	$\Pr(\mathrm{H} \geq 80) < 10^{-9}$

Getting much closer to the true probability.

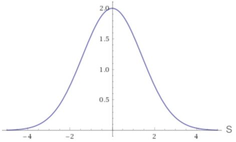
H: total number heads in 100 random coin flips. $\mathbb{E}[\mathrm{H}] = 50$.

Interpretation as a Central Limit Theorem

Bernstein Inequality (Simplified): Consider independent random variables $X_1, \ldots, X_n$ falling in [-1,1]. Let $\mu = \mathbb{E}[\sum X_i]$, $\sigma^2 = \text{Var}[\sum X_i]$, and $s \leq \sigma$. Then:

$$\Pr\left(\left|\sum_{i=1}^{n} X_i - \mu\right| \geq s\sigma\right) \leq 2\exp\left(-\frac{s^2}{4}\right).$$

Can plot this bound for different s:



Looks a lot like a Gaussian (normal) distribution.

$$\mathcal{N}(0, \sigma^2) \text{ has density } p(s\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{s^2}{2}}.$$

Gaussian Tails

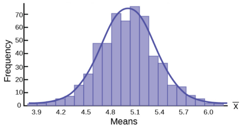
$$\mathcal{N}(0, \sigma^2) \text{ has density } p(s\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{s^2}{2}}.$$

Exercise: Using this can show that for $X \sim \mathcal{N}(0, \sigma^2)$: for any $s \geq 0$,

$$\Pr(|X| \geq s \cdot \sigma) \leq 2e^{-\frac{s^2}{2}}.$$

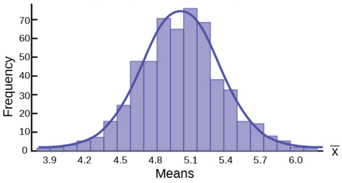
Essentially the same bound that Bernstein's inequality gives!

Central Limit Theorem Interpretation: Bernstein's inequality gives a quantitative version of the CLT. The distribution of the sum of *bounded* independent random variables can be upper bounded with a Gaussian (normal) distribution.



Central Limit Theorem

Stronger Central Limit Theorem: The distribution of the sum of n *bounded* independent random variables converges to a Gaussian (normal) distribution as n goes to infinity.



- Why is the Gaussian distribution is so important in statistics, science, ML, etc.?
- Many random variables can be approximated as the sum of a large number of small and roughly independent random effects. Thus, their distribution looks Gaussian by CLT.