COMPSCI 514: Algorithms for Data Science

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University of Massachusetts Amherst. Fall 2023.

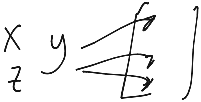
Lecture 11

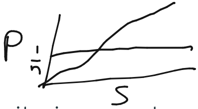
Logistics

- Problem Set 2 is due on Monday at 11:59pm.
- Midterm is in class Tuesday, 10/24. Thursday 10/19 will be devoted to midterm review.
- The grading on this week's quiz regarding the extra credit question had a bug. We will fix manually in the next few days.

Summary

Last Class:





- Locality sensitive hashing to solve the similarity search problem efficiently. S-curve
- MinHash as a locality sensitive hash function for Jaccard similarity.
- Brief look at SimHash as a locality sensitive hash function for cosine (dot product) similarity.

can a locality sensitive hash function be pairwise independent?

This Class:

- Introduce the frequent elements problem and its applications.
- Solution via the Count-Min sketch randomized data structure.

$$\Pr(h(x) = i \cap h(y) = j) = \frac{1}{n^2}$$

$$\Pr(h(x) = h(y)) = \frac{1}{n}$$

The Frequent Items Problems

k-**Frequent Items (Heavy-Hitters) Problem**: Consider a stream of n items $x_1, \ldots, x_n$ (with possible duplicates). Return any item at appears at least $\frac{n}{k}$ times.

$k=10$

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x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
5	12	3	3	4	5	5	10	3

***k*-Frequent Items (Heavy-Hitters) Problem**: Consider a stream of n items $x_1, \ldots, x_n$ (with possible duplicates). Return any item at appears at least $\frac{n}{k}$ times.

$k = 3$

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
5	12	3	3	4	5	5	10	3

all items that appeared $\frac{n}{k} = \frac{9}{3} = 3$ times

The Frequent Items Problems

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x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
5	12	3	3	4	5	5	10	3

- What is the maximum number of items that can be returned? a) n b) k c) n/k d) $\log n$

$\geq \frac{n}{k} \cdot k \geq n$

The Frequent Items Problems

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- What is the maximum number of items that can be returned? a) n $\boxed{\text{b) } k}$ c) n/k d) $\log n$
- Trivial with $O(n)$ space – store the count for each item and return the one that appears $\geq n/k$ times.
- Can we do it with less space? I.e., without storing all n items?

Applications of Frequent Items:

- Finding top/viral items (i.e., products on Amazon, videos watched on Youtube, Google searches, etc.)
- Finding very frequent IP addresses sending requests (to detect DoS attacks/network anomalies).
- 'Iceberg queries' for all items in a database with frequency above some threshold. $-\frac{n}{k}$

Generally want very fast detection, without having to scan through database/logs. I.e., want to maintain a running list of frequent items that appear in a stream.

Frequent Itemset Mining

Association rule learning: A very common task in data mining is to identify common associations between different events.

- Identified via frequent itemset counting. Find all sets of t items that appear many times in the same basket.
- Frequency of an itemset is known as its support.
- A single basket includes many different itemsets, and with many different baskets an efficient approach is critical. E.g., baskets are Twitter users and itemsets are subsets of who they follow.

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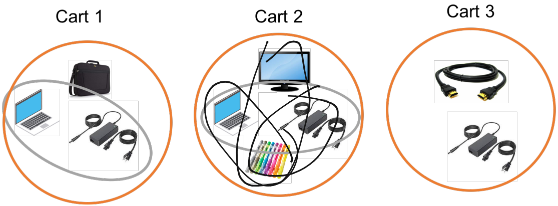

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Approximate Frequent Elements

Issue: No algorithm using $o(n)$ space can output just the items with frequency $\geq n/k$. Hard to tell between an item with frequency n/k (should be output) and $n/k - 1$ (should not be output).

x_1	x_2	x_3	x_4	x_5	x_6	...	$x_{n-n/k+1}$	...	x_n
3	12	9	27	4	101		3		3

n/k-1 occurrences

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n/k-1 occurrences

(ϵ, k)-Frequent Items Problem: Consider a stream of n items $x_1, \ldots, x_n$. Return a set F of items, including all items that appear at least $\frac{n}{k}$ times and only items that appear at least $(1-\epsilon) \cdot \frac{n}{k}$ times.

$k = 10 \Rightarrow$ return all items that show up at least $.1 \cdot n$ times

$\varepsilon = .1 \rightarrow$ should not return any item showing up $\leq (1-\varepsilon) \cdot .1 \cdot n = .09 \cdot n$ times

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- An example of relaxing to a 'promise problem': for items with frequencies in $[(1-\epsilon) \cdot \frac{n}{k}, \frac{n}{k}]$ no output guarantee.

Frequent Elements with Count-Min Sketch

Today: Count-min sketch – a random hashing based method closely related to bloom filters.

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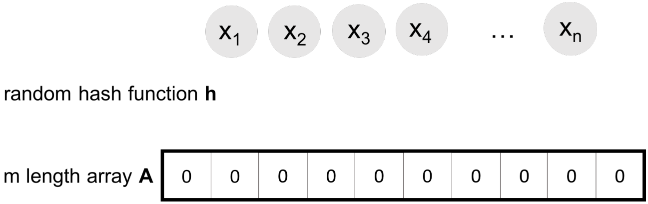

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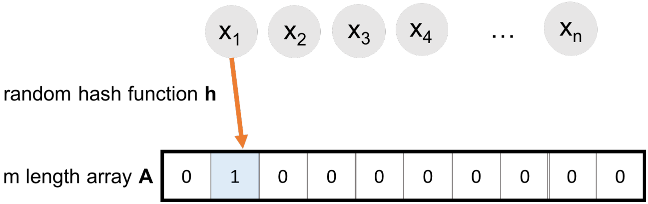

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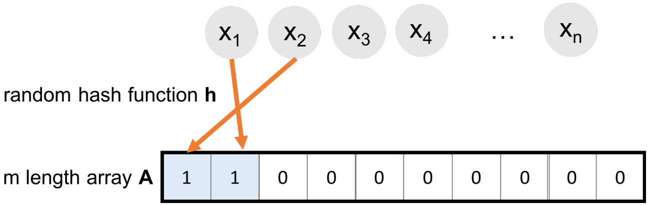

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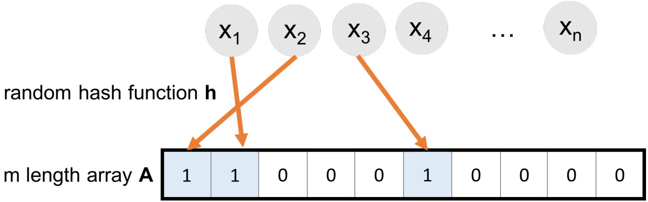

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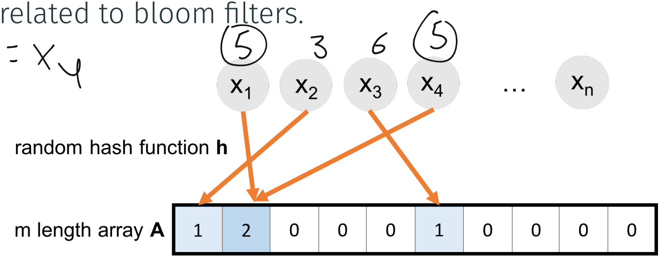
$X_1 = X_4$



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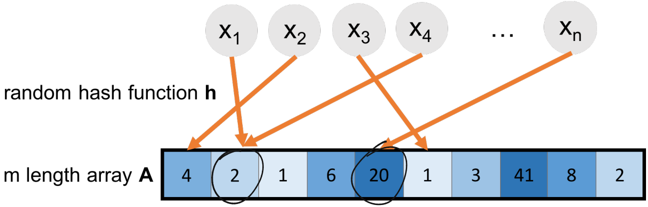

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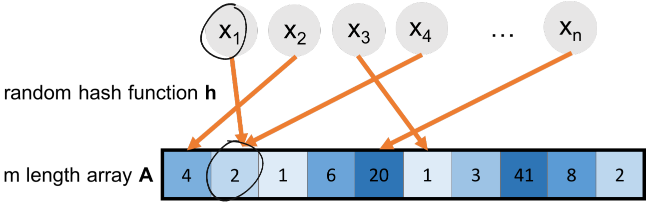


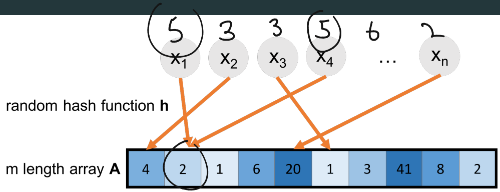
2

Will use $A[\mathbf{h}(x)]$ to estimate $f(x)$, the frequency of x in the stream. I.e., $|\{x_i : x_i = x\}|$.

Count-Min Sketch Accuracy

$h(5) = 2$

$A[2] \approx 2$



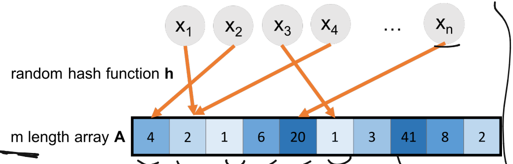
Use $A[\mathbf{h}(x)]$ to estimate $f(x)$.

Claim 1: We always have $A[\mathbf{h}(x)] \geq f(x)$. Why?

$f(x)$: frequency of x in the stream (i.e., number of items equal to x). $\mathbf{h}$: random hash function. m: size of Count-min sketch array.

$m << n$

$m \cdot \log(n)$



$\leq n$

Store m integers of size $\leq n$ this takes $m \cdot \log_2 n$ bits

Use $A[\mathbf{h}(x)]$ to estimate $f(x)$.

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- $A[\mathbf{h}(x)]$ counts the number of occurrences of any y with $\mathbf{h}(y) = \mathbf{h}(x)$, including x itself.

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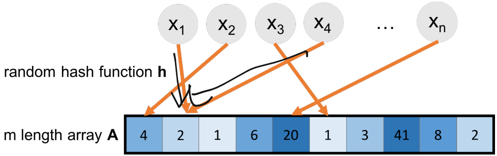

Use $A[\mathbf{h}(x)]$ to estimate $f(x)$.

Claim 1: We always have $A[\mathbf{h}(x)] \geq f(x)$. Why?

- $A[\mathbf{h}(x)]$ counts the number of occurrences of any y with $\mathbf{h}(y) = \mathbf{h}(x)$, including x itself.
- $A[\mathbf{h}(x)] = f(x) + \sum_{y \neq x : \mathbf{h}(y) = \mathbf{h}(x)} f(y)$.

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$$A[\mathrm{h}(x)] = f(x) + \underbrace{\sum_{y \neq x : \mathrm{h}(y) = \mathrm{h}(x)} f(y)}_{\text{error in frequency estimate}} .$$

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Expected Error:

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$\frac{1}{m}$

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$$= \sum_{y \neq x} \frac{1}{m} \cdot f(y) = \frac{1}{m} \cdot (n - f(x)) \leq \boxed{\frac{n}{m}}$$

$$\frac{1}{m} \sum_{y \neq x} f(y)$$

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What is a bound on probability that the error is $\geq \frac{2n}{m}$?

↳ Markov

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What is a bound on probability that the error is $\geq \frac{2n}{m}$?

Markov's inequality: $\Pr\left[\sum_{y \neq x : \mathbf{h}(y) = \mathbf{h}(x)} f(y) \geq \frac{2n}{m}\right] \leq \frac{1}{2}$.

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What property of $\mathbf{h}$ is required to show this bound? a) fully random b) pairwise independent c) 2-universal d) locality sensitive

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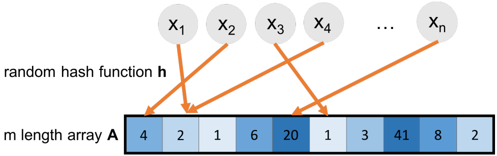
Count-Min Sketch Accuracy



Claim: For any x, with probability at least 1/2,

$$f(x) \leq A[\mathbf{h}(x)] \leq f(x) + \frac{2n}{m}.$$

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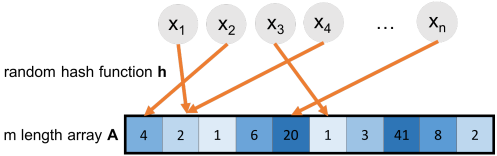

$\geq \frac{n}{k}$

$\leq \underline{(1-\epsilon)}\frac{n}{k}$

$\frac{2n}{m} = \frac{2n}{\frac{2k}{\epsilon}} = \epsilon \cdot \frac{n}{k}$

Claim: For any x, with probability at least 1/2,

$<(1-\epsilon)\frac{n}{k} + \epsilon\frac{n}{k} < \cancel{\frac{n}{k}}$

$$f(x) \leq A[\mathbf{h}(x)] \leq f(x) + \frac{2n}{m}.$$

To solve the (ϵ, k)-Frequent elements problem, set $m = \frac{2k}{\epsilon}$.

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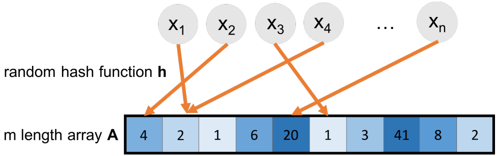

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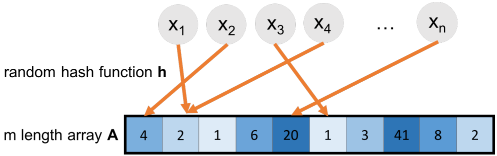

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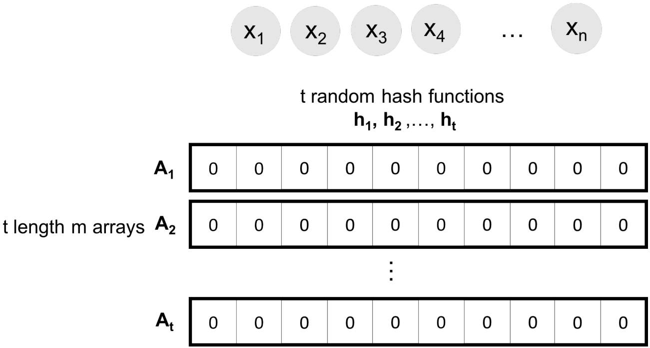
Count-Min Sketch Repetition



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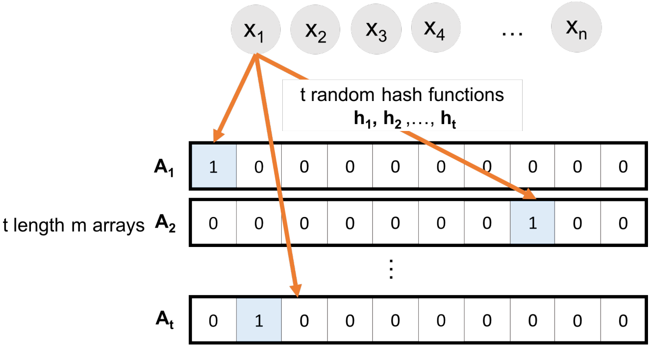

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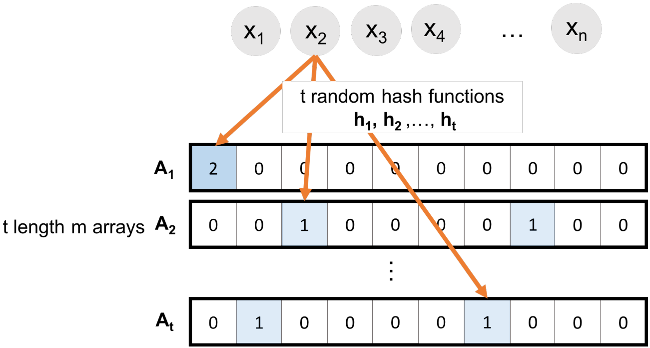

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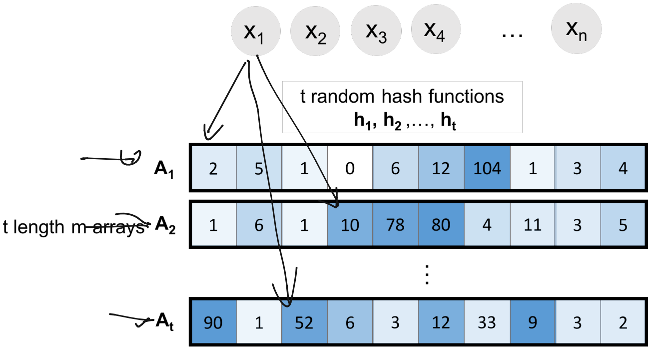

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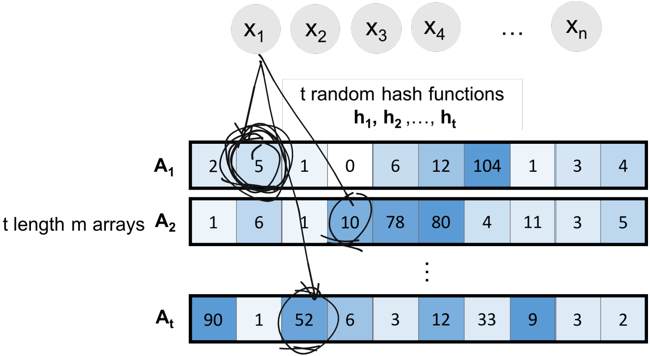
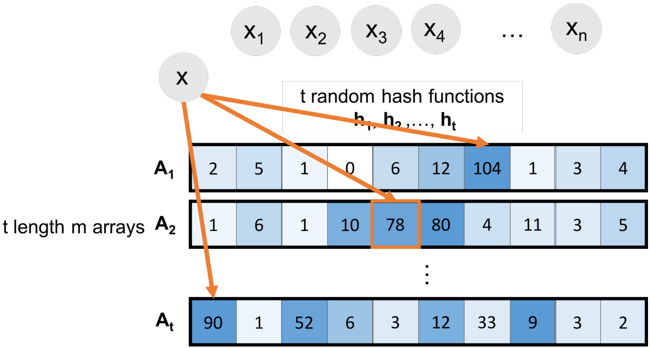


Estimate $f(x)$ with $\tilde{f}(x) = \min_{i \in [t]} A_i[\mathbf{h}_i(x)]$. (count-min sketch)

≈ 5

Count-Min Sketch Repetition



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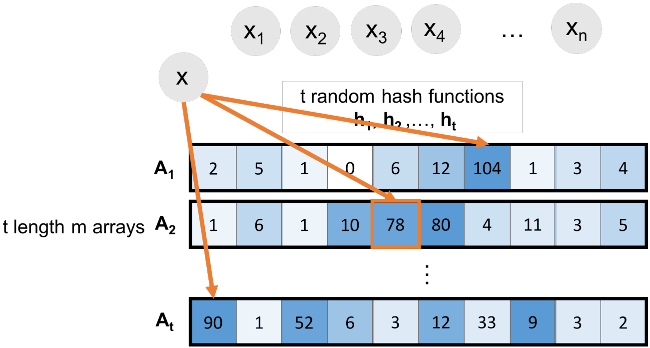

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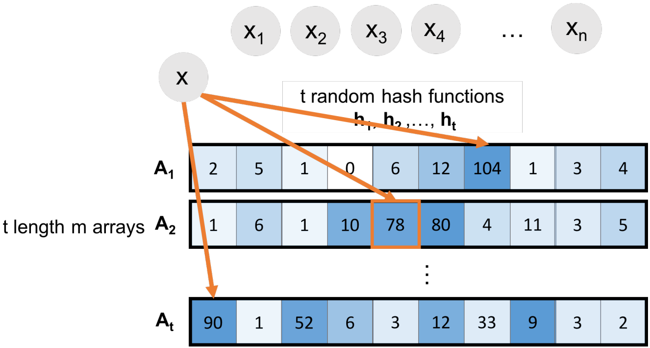
Why min instead of mean or median?

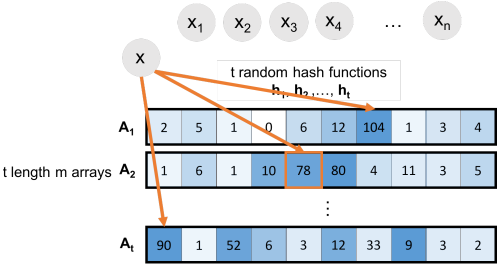
Count-Min Sketch Repetition



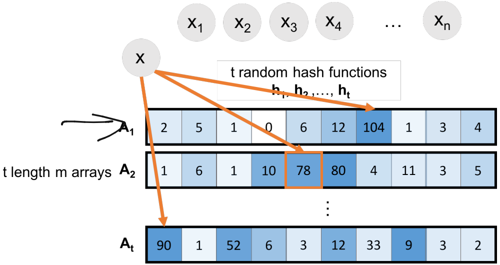
Estimate $f(x)$ with $\tilde{f}(x) = \min_{i \in [t]} A_i[\mathbf{h}_i(x)]$. (count-min sketch)

Why min instead of mean or median? The minimum estimate is always the most accurate since they are all overestimates of the true frequency!

Count-Min Sketch Analysis



Estimate $f(x)$ by $\tilde{f}(x) = \min_{i \in [t]} A_i[\mathbf{h}_i(x)]$



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- For every x and $i \in [t]$, we know that for $m = \frac{2k}{\epsilon}$, with probability $\geq 1/2$:

$$f(x) \leq A_i[\mathbf{h}_i(x)] \leq f(x) + \frac{\epsilon n}{k}.$$

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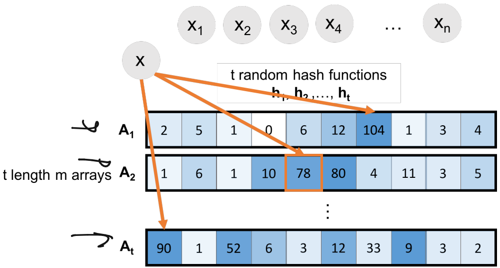

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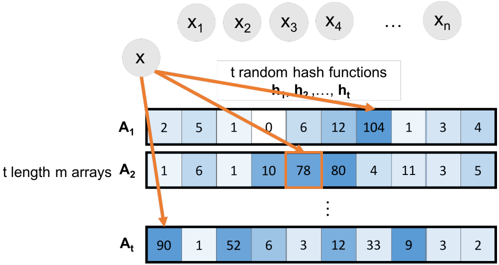

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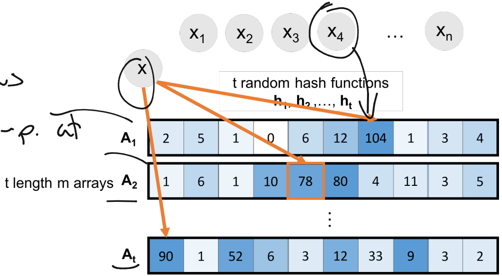
$\tilde{f}(x) \geq f(x) + \frac{\epsilon n}{k}$ only if $A_i[h(x)] \geq f(x) + \frac{\epsilon n}{k}$ for all t arrays

Count-Min Sketch Analysis

Via Markov
error $\leq \frac{\epsilon n}{k}$ w.p. at least 1/2



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$\geq 1-\delta$ $\quad$ $\leq \delta$

- What is $\Pr[f(x) \leq \tilde{f}(x) \leq f(x) + \frac{\epsilon n}{k}]$? $1 - 1/2^t$.
- To get a good estimate with probability $\geq 1 - \delta$, set $t = \log_2(1/\delta)$.

Upshot: Count-min sketch lets us estimate the frequency of every item in a stream up to error $\frac{\epsilon n}{k}$ with probability $\geq 1 - \delta$ in $O\left(\log(1/\delta) \cdot k/\epsilon\right)$ space.

<< n

$\cdot \log n$

m = length of each array

arrays

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- Accurate enough to solve the (ϵ, k)-Frequent elements problem – distinquish between items with frequency $\frac{n}{k}$ and those with frequency $(1-\epsilon)\frac{n}{k}$.

Count-Min Sketch

E_i event that estimate for item i is bad

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- Accurate enough to solve the (ϵ, k)-Frequent elements problem – distinquish between items with frequency $\frac{n}{k}$ and those with frequency $(1 - \epsilon)\frac{n}{k}$.
- How should we set δ if we want a good estimate for all items at once, with 99% probability? $\log(1/\delta) = \log(100n) = O(\log n)$

union bound

$\delta = \frac{.01}{n}$

$\Pr(E_1 \cup E_2 \ldots \cup E_n) \leq n \cdot \Pr(E_i) \leq n \cdot \delta \leq .01$

Identifying Frequent Elements

Count-min sketch gives an accurate frequency estimate for every item in the stream. But how do we identify the frequent items without having to store/look up the estimated frequency for all elements in the stream?

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don't know n ahead of time. don't know $\frac{n}{k}$

One approach:

$\frac{k}{2}$

- When a new item comes in at step i, check if its estimated frequency is $\geq i/k$ and store it if so. 3 $\cancel{5}$ 3
- At step i remove any stored items whose estimated frequency drops below i/k.
- Store at most $O(k)$ items at once and have all items with frequency $\geq n/k$ stored at the end of the stream.

$(1-\varepsilon)\frac{i}{k}$ $(1-\varepsilon)\frac{n}{k}$, violate wp $\leq \delta$