COMPSCI 514: Algorithms for Data Science

Cameron Musco

University of Massachusetts Amherst. Fall 2023.

Lecture 10

Logistics

- Problem Set 2 is due Monday 10/16 at 11:59pm.
- The midterm is in class on Tuesday 10/24. Midterm study material will be posted shortly.
- We have a quiz this week, but not the next two weeks (due to the problem set and midterm).

Summary

Last Class:

- Discussion of practical algorithms for distinct items estimation (LogLog/HyperLogLog).
- Introduction of Jaccard similarity and the similarity research problem.

This Class:

- Locality sensitive hashing for fast similarity search.
- MinHash as a locality sensitive hash function for Jaccard similarity
- Balancing false positives and negatives with LSH signatures and repeated hash tables.

Search with Jaccard Similarity

$$J(A, B) = \frac{|A \cap B|}{|A \cup B|} = \frac{\#\text{ shared elements}}{\#\text{ total elements}}.$$

Want Fast Implementations For:

- **Near Neighbor Search:** Have a database of n sets/bit strings and given a set A, want to find if it has high Jaccard similarity to anything in the database. $\Omega(n)$ time with a linear scan.
- **All-pairs Similarity Search:** Have n different sets/bit strings and want to find all pairs with high Jaccard similarity. $\Omega(n^2)$ time if we check all pairs explicitly.

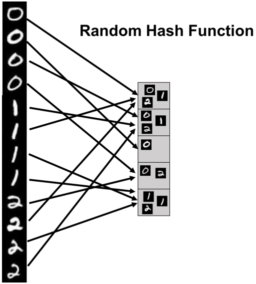
Will speed up via randomized locality sensitive hashing.

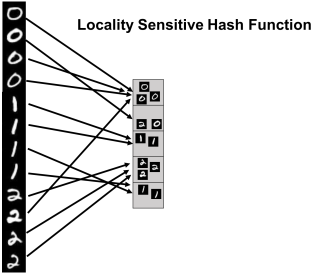
Locality Sensitive Hashing

Goal: Speed up Jaccard similarity search (near neighbor and all-pairs similarity search).

Strategy: Locality sensitive hashing (LSH).

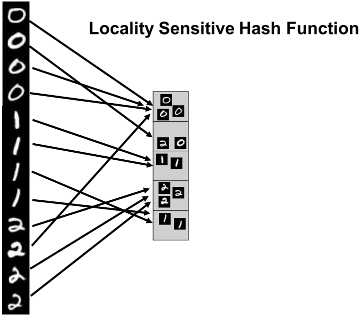
- Design a hash function where the collision probability is higher when two inputs are more similar (can design different functions for different similarity metrics.)





LSH For Similarity Search

How does locality sensitive hashing (LSH) help with similarity search?



- **Near Neighbor Search:** Given item x, compute $\mathbf{h}(x)$. Only search for similar items in the $\mathbf{h}(x)$ bucket of the hash table.
- **All-pairs Similarity Search:** Scan through all buckets of the hash table and look for similar pairs within each bucket.
- We will use $\mathbf{h}(x) = \mathbf{g}(MinHash(x))$ where $\mathbf{g} : [0,1] \to [n]$ is a random hash function. Why?

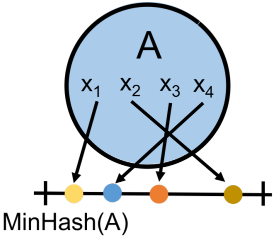
MinHashing

An Example: Locality sensitive hashing for Jaccard similarity.

Strategy: Use random hashing to map each set to a single hash value. The probably that two sets have colliding hash values will be proportional to their Jaccard similarity.

MinHash(A): [Andrei Broder, 1997 at Altavista]

- Let $\mathbf{h} : U \to [0, 1]$ be a random hash function
- $\mathbf{s} := 1$
- For $x_1, \ldots, x_{|A|} \in A$
    - $\mathbf{s} := \min(\mathbf{s}, \mathbf{h}(x_k))$
- Return $\mathbf{s}$



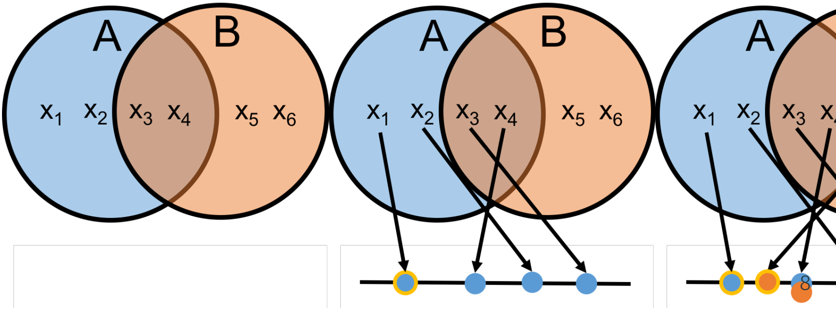
Identical to our distinct elements sketch!

MinHash Analysis

For two sets A and B, what is $\Pr(MinHash(A) = MinHash(B))$?

$$\Pr\left(\min_{x \in A} \mathbf{h}(x) = \min_{y \in B} \mathbf{h}(y)\right) = ?$$

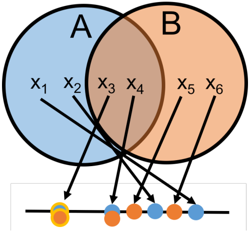
- Since we are hashing into the continuous range $[0, 1]$, we will never have $\mathbf{h}(x) = \mathbf{h}(y)$ for $x \neq y$ (i.e., no spurious collisions)



MinHash Analysis

For two sets A and B, what is $\Pr(MinHash(A) = MinHash(B))$?

Claim: $MinHash(A) = MinHash(B)$ only if an item in $A \cap B$ has the minimum hash value in both sets.



$$\Pr(MinHash(A) = MinHash(B)) =? \frac{|A \cap B|}{\text{total } \# \text{ items hashed}}$$

$$= \frac{|A \cap B|}{|A \cup B|} = J(A, B).$$

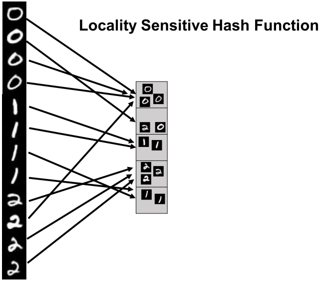
Locality sensitive: the higher $J(A, B)$ is, the more likely $MinHash(A), MinHash(B)$ are to collide.

Similarity Search with MinHash

Goal: Given a document y, identify all documents x in a database with Jaccard similarity (of their shingle sets) $J(x, y) \geq 1/2$.

Our Approach:

- Create a hash table of size m, choose a random hash function $\mathbf{g} : [0, 1] \to [m]$, and insert every item x into bucket $\mathbf{g}(MinHash(x))$. Search for items similar to y in bucket $\mathbf{g}(MinHash(y))$.



Reducing False Negatives

With a simple use of MinHash, we miss a match x with $J(x, y) = 1/2$ with probability 1/2. How can we reduce this false negative rate?

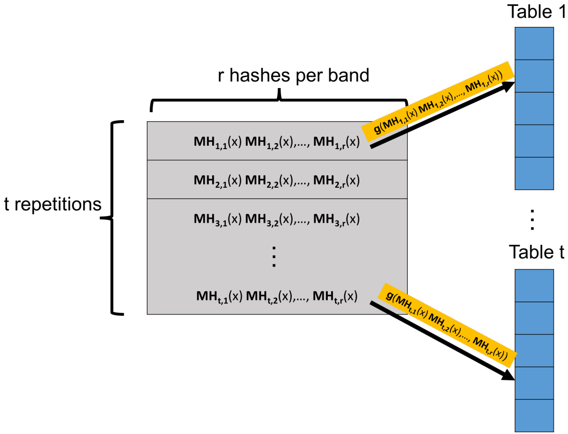
Repetition: Run MinHash t times independently, to produce hash values $MH_1(x), \ldots, MH_t(x)$. Apply random hash function $\mathbf{g}$ to map all these values to locations in t hash tables.

- To search for items similar to y, look at all items in bucket $\mathbf{g}(MH_1(y))$ of the 1^{st} table, bucket $\mathbf{g}(MH_2(y))$ of the 2^{nd} table, etc.
- What is the probability that x with $J(x, y) = 1/2$ is in at least one of these buckets, assuming for simplicity $\mathbf{g}$ has no collisions? $1-$ (probability in *no* buckets) $= 1 - \left(\frac{1}{2}\right)^t \approx .99$ for $t = 7$.
- What is the probability that x with $J(x, y) = 1/4$ is in at least one of these buckets, assuming for simplicity $\mathbf{g}$ has no collisions? $1-$ (probability in *no* buckets) $= 1 - \left(\frac{3}{4}\right)^t \approx .87$ for $t = 7$.

Potential for a lot of false positives! Slows down search time.

Balancing Hit Rate and Query Time

We want to balance a small probability of false negatives (a high hit rate) with a small probability of false positives (a small query time.)



Create t hash tables. Each is indexed into not with a single MinHash value, but with r values, appended together. A length r signature.

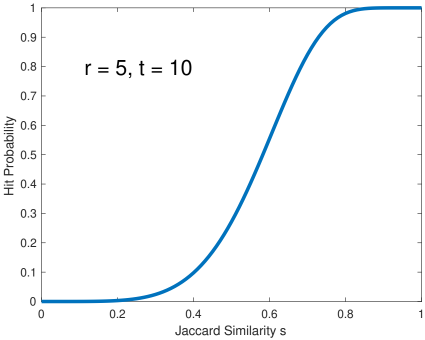
Balancing Hit Rate and Query Time

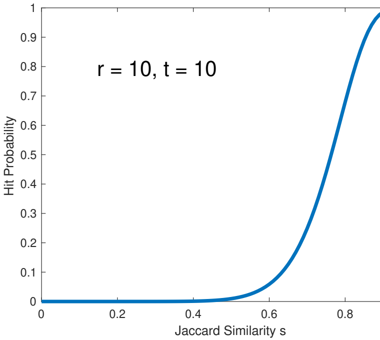
Consider searching for matches in t hash tables, using MinHash signatures of length r. For x and y with Jaccard similarity $J(x, y) = s$:

- Probability that a single hash matches. $\Pr\left[MH_{i,j}(x) = MH_{i,j}(y)\right] = J(x, y) = s$.
- Probability that x and y having matching signatures in repetition i. $\Pr\left[MH_{i,1}(x), \ldots, MH_{i,r}(x) = MH_{i,1}(y), \ldots, MH_{i,r}(y)\right] = s^r$.
- Probability that x and y don't match in repetition i: $1 - s^r$.
- Probability that x and y don't match in *all repetitions*: $(1 - s^r)^t$.
- Probability that x and y match in at least one repetition:

$$\text{Hit Probability: } 1 - (1 - s^r)^t.$$

Using t repetitions each with a signature of r MinHash values, the probability that x and y with Jaccard similarity $J(x, y) = s$ match in at least one repetition is: $1 - (1 - s^r)^t$.





r and t are tuned depending on application. 'Threshold' when hit probability is 1/2 is $\approx (1/t)^{1/r}$. E.g., $\approx (1/30)^{1/5} = .51$ in this case.

s-curve Example

For example: Consider a database with $10,000,000$ audio clips. You are given a clip x and want to find any y in the database with $J(x,y) \geq .9$.

- There are 10 true matches in the database with $J(x,y) \geq .9$.
- There are $10,000$ near matches with $J(x,y) \in [.7, .9]$.

With signature length $r = 25$ and repetitions $t = 50$, hit probability for $J(x,y) = s$ is $1 - (1 - s^{25})^{50}$.

- Hit probability for $J(x,y) \geq .9$ is $\geq 1 - (1 - .9^{25})^{50} \approx .98$
- Hit probability for $J(x,y) \in [.7, .9]$ is $\leq 1 - (1 - .9^{25})^{50} \approx .98$
- Hit probability for $J(x,y) \leq .7$ is $\leq 1 - (1 - .7^{25})^{50} \approx .007$

Expected Number of Items Scanned: (proportional to query time)

$$\leq 10 + .98 * 10,000 + .007 * 9,989,990 \approx 80,000 \ll 10,000,000.$$

Hashing for Duplicate Detection

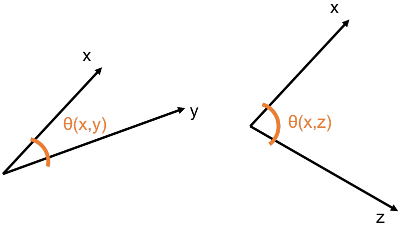
	Hash Table	Bloom Filters	MinHash Similarity Search	Distinct Elements
Goal	Check if x is a duplicate of any y in database and return y.	Check if x is a duplicate of y in database.	Check if x is a duplicate of any y in database and return y.	Count # of items, excluding duplicates.
Space	$O(n)$ items	$O(n)$ bits	$O(n \cdot t)$ items (when t tables used)	$O\left(\frac{\log \log n}{\epsilon^2}\right)$
Query Time	$O(1)$	$O(1)$	Potentially $o(n)$	NA
Approximate Duplicates?	✗	✗	✓	✗

All different variants of detecting duplicates/finding matches in large datasets. An important problem in many contexts!

Generalizing Locality Sensitive Hashing

Repetition and s-curve tuning can be used for fast similarity search with any similarity metric, given a locality sensitive hash function for that metric.

- LSH schemes exist for many similarity/distance measures: hamming distance, cosine similarity, etc.

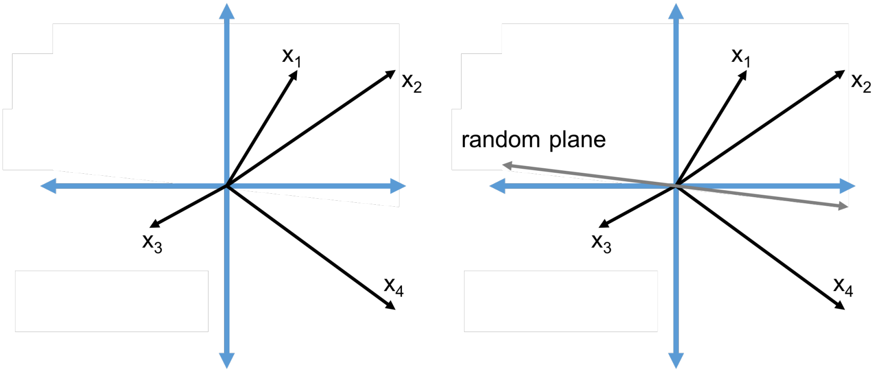


Cosine Similarity: $\cos(\theta(x,y)) = \frac{\langle x,y \rangle}{\|x\|_2 \cdot \|y\|_2}$.

- $\cos(\theta(x,y)) = 1$ when $\theta(x,y) = 0°$ and $\cos(\theta(x,y)) = 0$ when $\theta(x,y) = 90°$, and $\cos(\theta(x,y)) = -1$ when $\theta(x,y) = 180°$

SimHash for Cosine Similarity

SimHash Algorithm: LSH for cosine similarity.

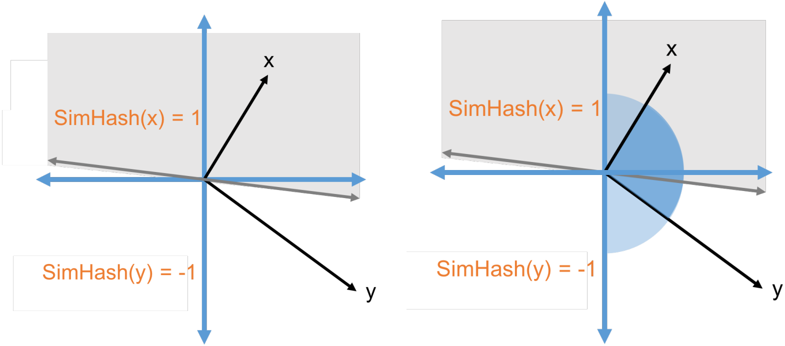


$SimHash(x) = \mathrm{sign}(\langle x, t\rangle)$ for a random vector t.

SimHash for Cosine Similarity

What is $\Pr\left[SimHash(x) = SimHash(y)\right]$?

$SimHash(x) \neq SimHash(y)$ when the plane separates x from y.



- $\Pr\left[SimHash(x) \neq SimHash(y)\right] = \frac{\theta(x,y)}{\pi}$
- $\Pr\left[SimHash(x) = SimHash(y)\right] = 1 - \frac{\theta(x,y)}{\pi} \approx \frac{\cos(\theta(x,y))+1}{2}$

Questions on MinHash and Locality Sensitive Hashing?