COMPSCI 514: Algorithms for Data Science

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University of Massachusetts Amherst. Fall 2022.
Lecture 8

Summary

Last Class:

- Finish up Bloom filter analysis and optimization of parameters.
- Start on streaming algorithms and distinct elements estimation via hashing.

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- Finish up Bloom filter analysis and optimization of parameters.
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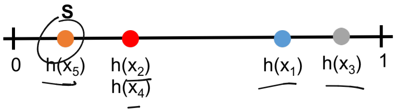
This Class:

- Analysis of the distinct elements algorithm.
- The median trick for boosting success probability.
- Sketch of the ideas behind practical algorithms for distinct elements estimation.

Hashing for Distinct Elements

Min-Hashing for Distinct Elements: 2 4 4 1 1 3 5 5

- Let $\mathbf{h} : U \to [0, 1]$ be a random hash function (with a real valued output)
- $s := 1$
- For $i = 1, \ldots, n$
  - $s := \min(s, \mathbf{h}(x_i))$
- Return $\tilde{d} = \frac{1}{s} - 1$



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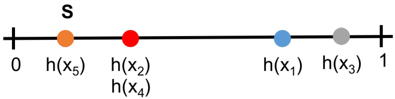

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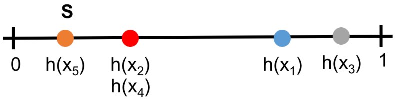

- After all items are processed, s is the minimum of d points chosen uniformly at random on $[0,1]$. Where $d = \#$ distinct elements.
- Intuition: The larger d is, the smaller we expect s to be.

Performance in Expectation

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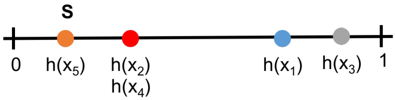
Where $d = \#$ distinct elements.



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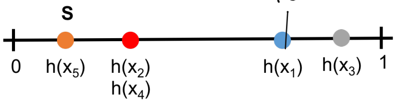
$\mathbb{E}s = \frac{1}{2}$

$d = 1$

$\mathbb{E}s = \frac{1}{3}$ $d = 2$

2/3



$d = 1$

$$\mathbb{E}[s] = \frac{1}{d+1}$$

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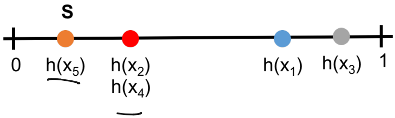

non-negative

$$\mathbb{E}[\mathbf{s}] = \frac{1}{d+1} \text{ (using } \mathbb{E}(\mathbf{s}) = \int_0^{\infty} \Pr(\mathbf{s} > x)dx) + \text{calculus)}$$

in range

$$\mathbb{E}X = \sum_s s \cdot \Pr(X = s)$$

$$= \sum_s \Pr(X \geq s)$$

in range

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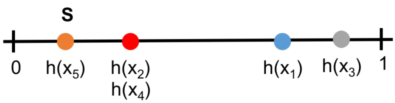

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$$\frac{1}{\frac{1}{d+1}} - 1 = d$$

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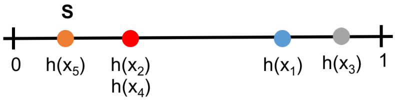

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$\mathbb{E}[s] = \frac{1}{d+1}$

$\frac{1}{\mathbb{E}[s]} - 1 = d$

$\mathbb{E}\left[\frac{1}{s} - 1\right] \neq \frac{1}{\mathbb{E}[s]} - 1$

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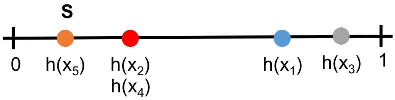

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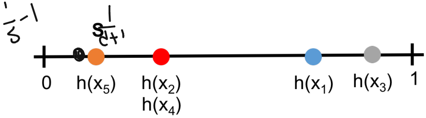

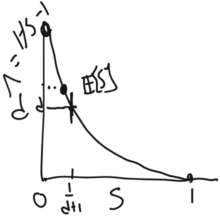

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$s \leq \frac{1}{d+1}(1+\varepsilon)$

$\hat{d} = \frac{1}{s} - 1 \geq \frac{1}{\frac{1}{(d+1)(1+\varepsilon)}} - 1$

$\geq (1-2\varepsilon)d$

- So our estimate $\hat{\mathbf{d}} = \frac{1}{\mathbf{s}} - 1$ is correct if **s** exactly equals its expectation. Does this mean $\mathbb{E}[\hat{\mathbf{d}}] = d$? No, but:
- **Approximation is robust:** if $|\mathbf{s} - \mathbb{E}[\mathbf{s}]| \leq \epsilon \cdot \mathbb{E}[\mathbf{s}]$ for any $\epsilon \in (0, 1/2)$ and a small constant $c \leq 4$:

$|\hat{d} - d| \leq < \varepsilon d$

$$(1 - c\epsilon)d \leq \hat{\mathbf{d}} \leq (1 + c\epsilon)d$$

$\hat{d} = \frac{1}{s} - 1$

Initial Concentration Bound

So question is how well s concentrates around its mean.

$$\mathbb{E}[\mathsf{s}] = \frac{1}{d+1}$$

s: minimum of d distinct hashes chosen randomly over $[0,1]$, computed by hashing algorithm. $\widehat{\mathsf{d}} = \frac{1}{\mathsf{s}} - 1$: estimate of $\#$ distinct elements d.

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Bound is vacuous for any $\epsilon < 1$.

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Leverage the law of large numbers: improve accuracy via repeated independent trials.

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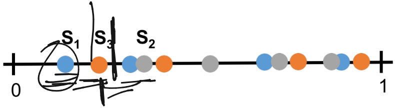
$\frac{1}{d+1} = \mathbb{E} S$

$k \cdot d$



Analysis

$s = \frac{1}{k}\sum_{j=1}^{k} s_j$. Have already shown that for $j = 1, \ldots, k$:

$$\mathbb{E}[s_j] = \frac{1}{d+1}$$

$$\mathrm{Var}[s_j] \leq \frac{1}{(d+1)^2}$$

s_j: minimum of d distinct hashes chosen randomly over $[0, 1]$. $s = \frac{1}{k}\sum_{j=1}^{k} s_j$.
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$$\mathrm{Var}[\mathsf{s}_j] \leq \frac{1}{(d+1)^2} \implies \mathrm{Var}[\mathsf{s}] \leq \frac{1}{k \cdot (d+1)^2} \text{ (linearity of variance)}$$

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$Var(s) = \frac{1}{k^2} \cdot \sum Var(s_j) = \frac{1}{k^2} \cdot k \cdot Var(s_j)$

$$\mathbb{E}[s_j] = \frac{1}{d+1} \implies \mathbb{E}[s] = \frac{1}{d+1} \text{ (linearity of expectation)}$$

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How should we set k if we want an error with probability at most δ?

$$k \geq \frac{1}{\epsilon^2 \delta} \qquad \frac{1}{.05^2 \cdot .01}$$

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How should we set k if we want an error with probability at most δ?

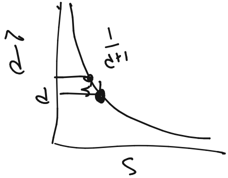
$k = \frac{1}{\epsilon^2 \cdot \delta}$.

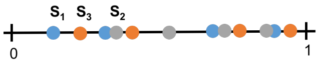
s_j: minimum of d distinct hashes chosen randomly over $[0,1]$. $\mathsf{s} = \frac{1}{k}\sum_{j=1}^{k} \mathsf{s}_j$.
$\widehat{\mathsf{d}} = \frac{1}{\mathsf{s}} - 1$: estimate of $\#$ distinct elements d.

Space Complexity

Hashing for Distinct Elements:

- Let $\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_k : U \to [0,1]$ be random hash functions
- $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_k := 1$
- For $i = 1, \ldots, n$
  - For j=1,..., k, $\mathbf{s}_j := \min(\mathbf{s}_j, \mathbf{h}_j(x_i))$
- $\mathbf{s} := \frac{1}{k}\sum_{j=1}^{k} \mathbf{s}_j$
- Return $\widehat{\mathbf{d}} = \frac{1}{\mathbf{s}} - 1$





- Setting $k = \frac{1}{\epsilon^2 \cdot \delta}$, algorithm returns $\widehat{\mathbf{d}}$ with $|d - \widehat{\mathbf{d}}| \leq 4\epsilon \cdot d$ with probability at least $1 - \delta$.

ε = relative error

δ = failure prob.

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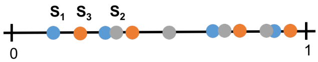

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d bits

- Space complexity is $k = \frac{1}{\epsilon^2 \cdot \delta}$ real numbers $s_1, \ldots, s_k$.

$\varepsilon = \delta = .01$ $\qquad \frac{1}{.01^3} = 1$ mil. $\qquad \frac{1}{.01^2} \cdot 5 = 50{,}000$

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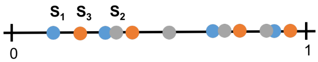
$E\, S = \frac{1}{d+1}$



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- $\delta = 5\%$ failure rate gives a factor 20 overhead in space complexity.

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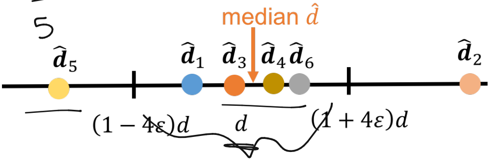
$\frac{1}{5}$



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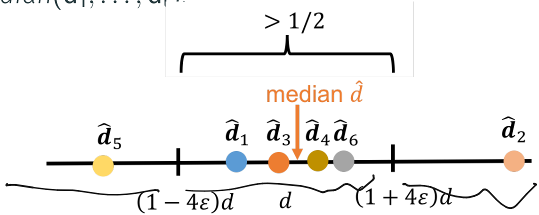

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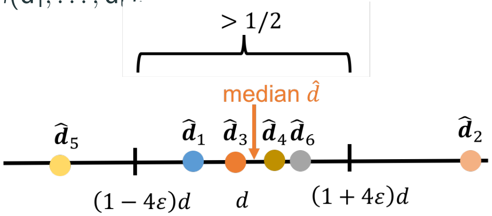

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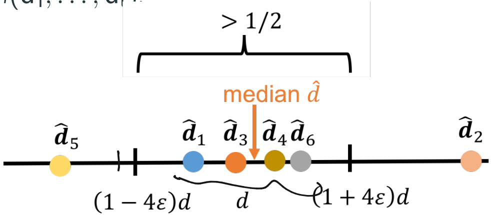
$1 - \delta' = 4/5$



- If $> 2/3$ of trials fall in $[(1-4\epsilon)d, (1+4\epsilon)d]$, then the median will.
- Have $< 1/3$ of trials on both the left and right.

The Median Trick

- $\widehat{\mathrm{d}}_1, \ldots, \widehat{\mathrm{d}}_t$ are the outcomes of the t trials, each falling in $[(1-4\epsilon)d, (1+4\epsilon)d]$ with probability at least 4/5.
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$\frac{5}{6} \cdot \frac{4}{5} = \frac{2}{3}$

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- Setting $t = O(\log(1/\delta))$ gives failure probability $e^{-\log(1/\delta)} = \delta$.

Median Trick

Upshot: The median of $t = O(\log(1/\delta))$ independent runs of the hashing algorithm for distinct elements returns $\widehat{d} \in [(1-4\epsilon)d, (1+4\epsilon)d]$ with probability at least $1-\delta$.

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A note on the median: The median is often used as a robust alternative to the mean, when there are outliers (e.g., heavy tailed distributions, corrupted data).

Distinct Elements in Practice

Our algorithm uses continuous valued fully random hash functions.

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The more distinct hashes we see, the higher we expect this maximum to be.

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With d distinct elements, roughly what do we expect **m** to be?

a) $O(1)$ b) $O(\log d)$ c) $O(\sqrt{d})$ d) $O(d)$

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So with d distinct hashes, expect to see 1 with $\log d$ trailing zeros.

Expect $\mathbf{m} \approx \log d$.

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Expect $\mathbf{m} \approx \log d$. **m** takes $\underline{\log \log d}$ bits to store.

LogLog Counting of Distinct Elements

Flajolet-Martin (LogLog) algorithm and HyperLogLog.

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Note: Careful averaging of estimates from multiple hash functions.

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- Set the maximum # of trailing zeros to the maximum in the two sketches.

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HyperLogLog In Practice

Implementations: Google PowerDrill, Facebook Presto, Twitter Algebird, Amazon Redshift.

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Traditional *COUNT*, *DISTINCT* SQL calls are far too slow, especially when the data is distributed across many servers.

Questions on distinct elements counting?