COMPSCI 514: Algorithms for Data Science

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University of Massachusetts Amherst. Fall 2022.

Lecture 5

Logistics

- Problem Set 1 is due this Friday at 11:59pm.
- Quiz question on class pacing:
  - Way too fast: 5.
  - A bit too fast: 45.
  - Just right: 53.
  - A bit too slow: 2.
  - Way too slow: 0.

Last Time

Last Class:

- 2-universal and pairwise independent hash functions.
- Chebyshev's inequality and the law of large numbers.
- The union bound.
- Application to hashing for load balancing.

X, X^2

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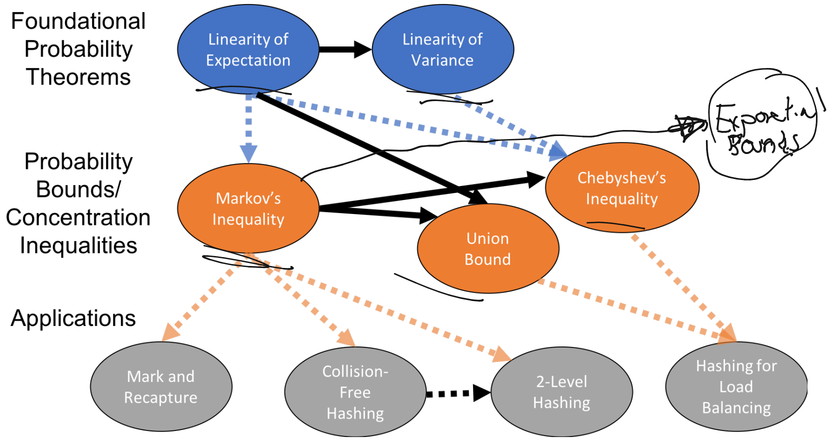
This Time:

- Exponential concentration bounds and the central limit theorem.

Concept Map



Foundational Probability Theorems
Probability Bounds/ Concentration Inequalities
Applications
Linearity of Expectation
Linearity of Variance
Markov's Inequality
Chebyshev's Inequality
Union Bound
Exponential Bounds
Mark and Recapture
Collision-Free Hashing
2-Level Hashing
Hashing for Load Balancing

Quiz Questions

$X_1, \ldots, X_5$ $\quad \mathbb{E}\sum X_i = \sum \mathbb{E} X_i = 5 \cdot .02 = .1$

$X = \sum X_i$

My (not very popular) photo hosting service receives ~~5~~ 2000 download requests per day. Each download request is completed successfully with probability 0.98. Give an upper bound on the probability that my service fails to complete at least one request successfully. Hint: do not assume independence of the request completions.

Answer:

Check

$1 - .98^5$

$\Pr(X \geq 1) \leq \frac{\mathbb{E}[X]}{1} \leq .1$

$\frac{.1}{2}$

A_i request i fails

$\Pr(A_1 \text{ or } A_2 \text{ or } \ldots A_5) \leq \sum_{i=1}^{5} \Pr(A_i)$

$= 5 \cdot .02 = .1$

$2000 \cdot .02 = 4$

Quiz Questions

$Pr(X \leq 65)$

$\leq Pr(|X - \mathbb{E}X| \geq 10)$

$\leq .12$

X_i

The expected temperature on Saturday is $\mu = 75$ degrees. The variance of the temperature is $\sigma^2 = 12$ degrees. Give an upper bound on the probability that the temperature ~~does not lie between~~ 60 65 and 85 degrees.

$Pr(|X - \mathbb{E}X| \geq 10) \leq .12$

Answer:

Check

μ

65 75 85

$$Pr(|X - \mathbb{E}X| \geq t) \leq \frac{Var(X)}{t^2}$$

$Pr(X \geq 85)$

$$Pr(|X - 75| \geq 10) \leq \frac{12}{10^2} = .12$$

$= Pr(X - \mathbb{E}X \geq 10)$

$\leq Pr(|X - \mathbb{E}X| \geq 10)$

Flipping Coins

$H_i = 1$ if coin i is heads, 0 otherwise

We flip $n = 100$ independent coins, each are heads with probability 1/2 and tails with probability 1/2. Let H be the number of heads.

$$\mathbb{E}[\mathsf{H}] = 50 \quad \text{and} \quad \mathrm{Var}[\mathsf{H}] = \sum_{i=1}^{100} \mathrm{Var}(H_i)$$

$p(1-p)$

$.5 \cdot .5 = .25$

$$= \sum_{i=1}^{100} .25 = \underline{25}$$

Flipping Coins

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$$\mathbb{E}[\mathsf{H}] = \frac{n}{2} = 50 \text{ and } \mathsf{Var}[\mathsf{H}] = \frac{n}{4} = 25$$

We flip $n = 100$ independent coins, each are heads with probability 1/2 and tails with probability 1/2. Let H be the number of heads.

Bin(n, p)

$n = 100$ $p = .5$

$$\mathbb{E}[\mathrm{H}] = \frac{n}{2} = 50 \text{ and } \mathrm{Var}[\mathrm{H}] = \frac{n}{4} = 25$$

Markov's:	Chebyshev's:	In Reality:
$\frac{\mathbb{E}[H]}{60} = \frac{50}{60} = 5/6$	$\frac{\mathrm{Var}(X)}{10^2}$	
$\Pr(\mathrm{H} \geq 60) \leq .833$	$\Pr(\mathrm{H} \geq 60) \leq .25$	$\Pr(\mathrm{H} \geq 60) = 0.0284$
$\Pr(\mathrm{H} \geq 70) \leq .714$	$\Pr(\mathrm{H} \geq 70) \leq .0625$	$\Pr(\mathrm{H} \geq 70) = .000039$
$\Pr(\mathrm{H} \geq 80) \leq .625$	$\Pr(\mathrm{H} \geq 80) \leq .0278$	$\Pr(\mathrm{H} \geq 80) < 10^{-9}$
5/8		

H has a simple Binomial distribution, so can compute these probabilities exactly.

To be fair.... Markov and Chebyshev's inequalities apply much more generally than to Binomial random variables like coin flips.

Can we obtain tighter concentration bounds that still apply to very general distributions?

Tighter Concentration Bounds

To be fair.... Markov and Chebyshev's inequalities apply much more generally than to Binomial random variables like coin flips.

Can we obtain tighter concentration bounds that still apply to very general distributions?

- Markov's: $\Pr(X \geq t) \leq \frac{\mathbb{E}[X]}{t}$. First Moment.

Tighter Concentration Bounds

To be fair.... Markov and Chebyshev's inequalities apply much more generally than to Binomial random variables like coin flips.

Can we obtain tighter concentration bounds that still apply to very general distributions?

- Markov's: $\Pr(X \geq t) \leq \frac{\mathbb{E}[X]}{t}$. First Moment.
- Chebyshev's: $\Pr(|X - \mathbb{E}[X]| \geq t) = \Pr(|X - \mathbb{E}[X]|^2 \geq t^2) \leq \frac{\text{Var}[X]}{t^2}$. Second Moment.

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To be fair.... Markov and Chebyshev's inequalities apply much more generally than to Binomial random variables like coin flips.

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- Markov's: $\Pr(X \geq t) \leq \frac{\mathbb{E}[X]}{t}$. First Moment.
- Chebyshev's: $\Pr(|X - \mathbb{E}[X]| \geq t) = \Pr(|X - \mathbb{E}[X]|^2 \geq t^2) \leq \frac{\mathrm{Var}[X]}{t^2}$. Second Moment.
- What if we just apply Markov's inequality to even higher moments?

A Fourth Moment Bound

Consider any random variable X:

$$\Pr(|X - \mathbb{E}[X]| \geq t) = \Pr\left((X - \mathbb{E}[X])^4 \geq t^4\right)$$

$$=$$

$$\Pr\left(|X - \mathbb{E}X|^3 \geq t^3\right)$$

A Fourth Moment Bound

Consider any random variable X:

$$\Pr(|X - \mathbb{E}[X]| \geq t) = \Pr\left((X - \mathbb{E}[X])^4 \geq t^4\right) \leq \frac{\mathbb{E}\left[(X - \mathbb{E}[X])^4\right]}{t^4}.$$

A Fourth Moment Bound

Consider any random variable X:

Markov's applied to $(X-EX)^4$

$$\Pr(|X - \mathbb{E}[X]| \geq t) = \Pr\left((X - \mathbb{E}[X])^4 \geq t^4\right) \leq \frac{\mathbb{E}\left[(X - \mathbb{E}[X])^4\right]}{t^4}.$$

Application to Coin Flips: Recall: $n = 100$ independent fair coins, H is the number of heads.

- Bound the fourth moment:

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- Bound the fourth moment:

$$\mathbb{E}\left[(H - \mathbb{E}[H])^4\right] = \mathbb{E}\left[\left(\sum_{i=1}^{100} H_i - 50\right)^4\right]$$

where $H_i = 1$ if coin flip i is heads and 0 otherwise.

A Fourth Moment Bound

$\frac{1}{2^4} = \frac{1}{16}$

Consider any random variable X:

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$$\mathbb{E}\left[(H - \mathbb{E}[H])^4\right] = \mathbb{E}\left[\left(\sum_{i=1}^{100} H_i - 50\right)^4\right] = \sum_{i,j,k,\ell} c_{ijk\ell}\mathbb{E}[H_i H_j H_k H_\ell]$$

$+ \sum \mathbb{E} H_i H_j H_k$

$+ C$

where $H_i = 1$ if coin flip i is heads and 0 otherwise. Then apply some messy calculations...

$(H_1 + H_2 + H_3 - 50)^4$

$H_1^4 + H_1^3 H_2 + \ldots \; H_1^3 \cdot -50$

Consider any random variable X:

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- Apply Fourth Moment Bound: $\Pr(\underline{|H - \mathbb{E}[H]|} \geq \underline{t}) \leq \frac{1862.5}{t^4}$.

$\Pr(H - \mathbb{E}[H] > t) \leq \frac{25}{t^2}$

Tighter Bounds

$Pr(H \geq 51)$

$\frac{1862.5}{10^4}$

Chebyshev's:	4^{th} Moment:	In Reality:
$\Pr(H \geq 60) \leq .25$	$\Pr(H \geq 60) \leq .186$	$\Pr(H \geq 60) = 0.0284$
$\Pr(H \geq 70) \leq .0625$	$\Pr(H \geq 70) \leq .0116$	$\Pr(H \geq 70) = .000039$
$\Pr(H \geq 80) \leq .04$	$\Pr(H \geq 80) \leq .0023$	$\Pr(H \geq 80) < 10^{-9}$

$\frac{25}{1^2}$ $\frac{1862.5}{1^4}$

H: total number heads in 100 random coin flips. $\mathbb{E}[H] = 50$.

Tighter Bounds

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Can we just keep applying Markov's inequality to higher and higher moments and getting tighter bounds?

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Tighter Bounds

Chebyshev's:	4^{th} Moment: $\frac{1762.5}{t^4}$	In Reality:
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- Yes! To a point.
- In fact – don't need to just apply Markov's to $|X - \mathbb{E}[X]|^k$ for some k. Can apply to any monotonic function $f(|X - \mathbb{E}[X]|)$.

monotonically increasing for non-negative inputs

$$\Pr(|X - \mathbb{E}X| \geq t) = \Pr(f(|X - \mathbb{E}X|) \geq f(t)) \leq \frac{\mathbb{E}[f(|X - \mathbb{E}X|)]}{f(t)}$$

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- Why monotonic?

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4.5

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- In fact – don't need to just apply Markov's to $|X - \mathbb{E}[X]|^k$ for some k. Can apply to any monotonic function $f(|X - \mathbb{E}[X]|)$.
- Why monotonic? $\Pr(|X - \mathbb{E}[X]| > t) = \Pr(f(|X - \mathbb{E}[X]|) > f(t))$.

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Exponential Concentration Bounds

Moment Generating Function: Consider for any $t > 0$:

$$M_t(X) = e^{t\cdot(X - \mathbb{E}[X])}$$

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- Choosing t appropriately lets one prove a number of very powerful exponential concentration bounds (exponential tail bounds).

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- Chernoff bound, Bernstein inequalities, Hoeffding's inequality, Azuma's inequality, Berry-Esseen theorem, etc.

Exponential Concentration Bounds

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- Weighted sum of all moments, with t controlling how slowly the weights fall off (larger t = slower falloff).
- Choosing t appropriately lets one prove a number of very powerful exponential concentration bounds (exponential tail bounds).
- Chernoff bound, Bernstein inequalities, Hoeffding's inequality, Azuma's inequality, Berry-Esseen theorem, etc.
- We will not cover the proofs in this class.

Bernstein Inequality

$\{0,1\} \in [-1,1]$ $\quad M=1$

Bernstein Inequality: Consider independent random variables $X_1, \ldots, X_n$ all falling in $[-M, M]$. Let $\mu = \mathbb{E}[\sum_{i=1}^n X_i]$ and $\sigma^2 = \text{Var}[\sum_{i=1}^n X_i] = \sum_{i=1}^n \text{Var}[X_i]$. For any $t \geq 0$:

$$\Pr\left(\left|\sum_{i=1}^n X_i - \mu\right| \geq t\right) \leq 2\exp\left(-\frac{t^2}{2\sigma^2 + \frac{4}{3}Mt}\right).$$

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$e^{-.001} \approx 1$

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Assume that $M = 1$ and plug in $t = s \cdot \sigma$ for $s \leq \sigma$.

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Compare to Chebyshev's: $\Pr\left(\left|\sum_{i=1}^n X_i - \mu\right| \geq s\sigma\right) \leq \frac{1}{s^2}$.

$\frac{Var(X)}{s^2\sigma^2} = \frac{1}{s^2}$

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Assume that $M = 1$ and plug in $t = s \cdot \sigma$ for $s \leq \sigma$.

Compare to Chebyshev's: $\Pr\left(\left|\sum_{i=1}^n X_i - \mu\right| \geq s\sigma\right) \leq \frac{1}{s^2}$.

- An exponentially stronger dependence on s!

Comparision to Chebyshev's

Consider again bounding the number of heads H in $n = 100$ independent coin flips.

Chebyshev's:	Bernstein:	In Reality:
$\Pr(H \geq 60) \leq .25$	$\Pr(H \geq 60) \leq .21$	$\Pr(H \geq 60) = 0.0284$
$\Pr(H \geq 70) \leq .0625$	$\Pr(H \geq 70) \leq .005$	$\Pr(H \geq 70) = .000039$
$\Pr(H \geq 80) \leq .04$	$\Pr(H \geq 80) \leq 4^{-5}$	$\Pr(H \geq 80) < 10^{-9}$

H: total number heads in 100 random coin flips. $\mathbb{E}[H] = 50$.

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Getting much closer to the true probability.

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