COMPSCI 514: Algorithms for Data Science

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University of Massachusetts Amherst. Fall 2022.

Lecture 4

Logistics

- Problem Set 1 due next Friday 9/23, at 11:59pm.
- Second quiz will be released today after class, due Monday 8:00pm.
- I will hold additional office hours next Tuesday 11am-12pm.

Last Time

Last Class:

- Expected collision analysis for hashing and collision free hashing via Markov's inequality. Gives $O(1)$ query time and $O(m^2)$ space for item look-up problem.
- 2-level hashing and its analysis via linearity of expectation. Gives optimal $O(1)$ query time and $O(m)$ space.

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This Time:

- 2-universal and pairwise independent hash functions
- Hashing for load balancing. Motivating:
    - Stronger concentration inequalities: Chebyshev's inequality, exponential tail bounds, and their connections to the law of large numbers and central limit theorem.
    - The union bound to bound the probability that one of multiple possible correlated events happens.

Efficiently Computable Hash Function

So Far: we have assumed a **fully random hash function** $\mathbf{h}(x)$ with $\Pr[\mathbf{h}(x) = i] = \frac{1}{n}$ for $i \in 1, \ldots, n$ and $\mathbf{h}(x), \mathbf{h}(y)$ independent for $x \neq y$.

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- To compute a random hash function we have to store a table of x values and their hash values. Would take at least $O(m)$ space and $O(m)$ query time to look up $\mathbf{h}(x)$ if we hash m values. Making our whole quest for $O(1)$ query time pointless!

h(x) :
output rand(1,..n);
end

x	h(x)
x_1	45
x_2	1004
x_3	10
⋮	⋮
x_m	12

Efficiently Computable Hash Functions

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2-Universal Hash Function (low collision probability). A random hash function from $\mathbf{h} : U \to [n]$ is two universal if:

$$\Pr[\mathbf{h}(x) = \mathbf{h}(y)] \leq \frac{1}{n}.$$

$\Pr[h(1) = h(2)] = 0$

$\Pr[h(1) = h(j)] = \frac{1}{n}$

$j > n$

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Exercise: Rework the two level hashing proof to show that this property is really all that is needed.

O(m) O(1)

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When $\mathbf{h}(x)$ and $\mathbf{h}(y)$ are chosen independently at random from $[n]$, $\Pr[\mathbf{h}(x) = \mathbf{h}(y)] = \frac{1}{n}$ (so a fully random hash function is 2-universal)

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Efficient Alternative: Let p be a prime with $p \geq |U|$. Choose random $\mathbf{a}, \mathbf{b} \in [p]$ with $\mathbf{a} \neq 0$. Represent x an an integer and let

$$\mathbf{h}(x) = (\mathbf{a}x + \mathbf{b} \mod p) \mod n.$$

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$$\Pr[\mathbf{h}(x) = i \cap \mathbf{h}(y) = j] = \frac{1}{n^2}.$$

$$Pr(h(x) = i) \cdot Pr(h(y) = j)$$

$$\frac{1}{n} \cdot \frac{1}{n} = \frac{1}{n^2}$$

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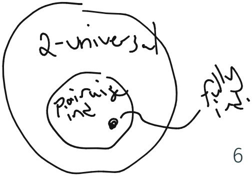
$$\Pr[\mathbf{h}(x) = \mathbf{h}(y)] = \sum_{i=1}^{n} \Pr[\mathbf{h}(x) = i \cap \mathbf{h}(y) = i] = n \cdot \frac{1}{n^2} = \frac{1}{n}.$$



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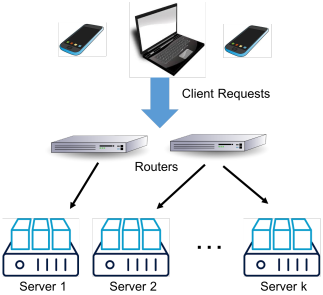
A closely related $(\mathbf{a}x + \mathbf{b}) \mod p$ construction gives pairwise independence on top of 2-universality.

Remember: A fully random hash function is both 2-universal and pairwise independent. But it is not efficiently implementable.

Another Application

Randomized Load Balancing:



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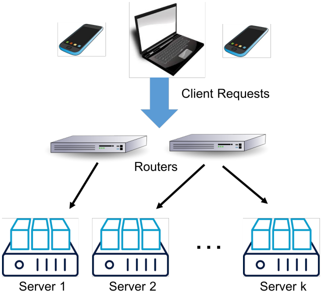

Simple Model: n requests randomly assigned to k servers. How many requests must each server handle?

- Often assignment is done via a random hash function. Why?

Weakness of Markov's

$\mathbb{E}[R_i] = \frac{n}{k}$

n requests

k servers

R_i = # requests on server i

n: total number of requests, k: number of servers randomly assigned requests, R_i: number of requests assigned to server i.

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If we provision each server be able to handle twice the expected load, what is the probability that a server is overloaded?

$\frac{2n}{k}$

$\Pr(R_i \geq 2\mathbb{E}[R_i]) \leq \frac{1}{2}$

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Applying Markov's Inequality

$$\Pr[R_i \geq 2\mathbb{E}[R_i]] \leq \frac{\mathbb{E}[R_i]}{2\mathbb{E}[R_i]} = \frac{1}{2}.$$

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$$\Pr[R_i \geq 2\mathbb{E}[R_i]] \leq \frac{\mathbb{E}[R_i]}{2\mathbb{E}[R_i]} = \frac{1}{2}.$$

Not great...half the servers may be overloaded.

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Chebyshev's inequality:

$\mathbb{E}[(X - \mathbb{E}X)^2]$

$\forall t,$

$$\Pr(|X - \mathbb{E}[X]| \geq t) \leq \frac{\mathrm{Var}[X]}{t^2}.$$

(by plugging in the random variable $X - \mathbb{E}[X]$)

Chebyshev's inequality

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X: any random variable, t, s: any fixed numbers.

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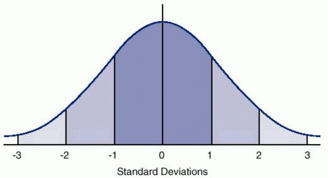
What is the probability that X falls s standard deviations from it's mean?



$$\Pr\left(|X - \mathbb{E}X| \geq s \cdot \sqrt{\mathrm{Var}(X)}\right) \leq \frac{\mathrm{Var}(X)}{s^2 \cdot \mathrm{Var}(X)} = \frac{1}{s^2}$$

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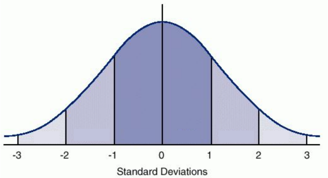

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Law of Large Numbers

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$\mathbb{E}S = \mu$

$|S - \mu|$

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$$\text{Var}[S] = \text{Var}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n^2}\text{Var}\left(\sum_{i=1}^{n} X_i\right)$$

linearity of variance

$$= \frac{1}{n^2}\sum_{i=1}^{n}\text{Var}(X_i)$$

$$= \frac{1}{n^2}\cdot\sum_{i=1}^{n}\sigma^2 = \frac{1}{n^2}\cdot n\cdot\sigma^2 = \frac{\sigma^2}{n}$$

$$\frac{1}{n-1}\sum(X_i - S)^2$$

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By Chebyshev's Inequality: for any fixed value $\epsilon > 0$,

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$\mathbb{E} X_1$

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$\mathbb{E} S = \frac{1}{n}\sum \mathbb{E} X_i = \frac{1}{n}\cdot n\cdot\mu = \mu$

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- Cannot show from vanilla Markov's inequality.

Load Balancing Variance

We can write the number of requests assigned to server i, R_i as:

$$R_i = \sum_{j=1}^{n} R_{i,j}$$

where $R_{i,j}$ is 1 if request j is assigned to server i and 0 otherwise.

n: total number of requests, k: number of servers randomly assigned requests, R_i: number of requests assigned to server i.

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$$\text{Var}[R_i] = \sum_{j=1}^{n} \text{Var}[R_{i,j}] \qquad \text{(linearity of variance)}$$

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where $R_{i,j}$ is 1 if request j is assigned to server i and 0 otherwise.

$$\mathrm{Var}[R_{i,j}] = \mathbb{E}\left[\left(R_{i,j} - \mathbb{E}[R_{i,j}]\right)^2\right] = \frac{1}{k}\left(1-\frac{1}{k}\right)^2 + \left(1-\frac{1}{k}\right)\left(0-\frac{1}{k}\right)^2$$

$$\mathbb{E}[R_{i,j}] = 1/k$$

$$R_{i,j} = \begin{cases} 1 & \text{w.p. } \frac{1}{k} \\ 0 & \text{w.p. } 1-\frac{1}{k} \end{cases}$$

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where $\mathsf{R}_{i,j}$ is 1 if request j is assigned to server i and 0 otherwise.

$$\begin{aligned}\mathsf{Var}[\mathsf{R}_{i,j}] &= \mathbb{E}\left[\left(\mathsf{R}_{i,j} - \mathbb{E}[\mathsf{R}_{i,j}]\right)^2\right] \\ &= \Pr(\mathsf{R}_{i,j} = 1) \cdot \left(1 - \mathbb{E}[\mathsf{R}_{i,j}]\right)^2 + \Pr(\mathsf{R}_{i,j} = 0) \cdot \left(0 - \mathbb{E}[\mathsf{R}_{i,j}]\right)^2\end{aligned}$$

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\text{Var}[\mathsf{R}_{i,j}] &= \mathbb{E}\left[\left(\mathsf{R}_{i,j} - \mathbb{E}[\mathsf{R}_{i,j}]\right)^2\right] \\
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&= \frac{1}{k} \cdot \left(1 - \frac{1}{k}\right)^2 + \left(1 - \frac{1}{k}\right) \cdot \left(0 - \frac{1}{k}\right)^2
\end{aligned}$$

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$R_i = \sum_{j=1}^{n} R_{ij} \Rightarrow$

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where $R_{i,j}$ is 1 if request j is assigned to server i and 0 otherwise.

$$\begin{aligned}
\text{Var}[R_{i,j}] &= \mathbb{E}\left[\left(R_{i,j} - \mathbb{E}[R_{i,j}]\right)^2\right] \\
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&= \frac{1}{k} \cdot \left(1 - \frac{1}{k}\right)^2 + \left(1 - \frac{1}{k}\right) \cdot \left(0 - \frac{1}{k}\right)^2 \\
&= \frac{1}{k} - \frac{1}{k^2} \leq \frac{1}{k}
\end{aligned}$$

$\frac{1}{k}\left(1 - \frac{1}{k}\right)$

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$\sum_{j=1} \text{Var}(R_{i,j}) \leq n \cdot \frac{1}{k}$

where $R_{i,j}$ is 1 if request j is assigned to server i and 0 otherwise.

$$\begin{aligned}
\text{Var}[R_{i,j}] &= \mathbb{E}\left[\left(R_{i,j} - \mathbb{E}[R_{i,j}]\right)^2\right] \\
&= \Pr(R_{i,j} = 1) \cdot \left(1 - \mathbb{E}[R_{i,j}]\right)^2 + \Pr(R_{i,j} = 0) \cdot \left(0 - \mathbb{E}[R_{i,j}]\right)^2 \\
&= \frac{1}{k} \cdot \left(1 - \frac{1}{k}\right)^2 + \left(1 - \frac{1}{k}\right) \cdot \left(0 - \frac{1}{k}\right)^2 \\
&= \frac{1}{k} - \frac{1}{k^2} \leq \frac{1}{k} \implies \text{Var}[R_i] \leq \frac{n}{k}.
\end{aligned}$$

n: total number of requests, k: number of servers randomly assigned requests, R_i: number of requests assigned to server i.

Bounding the Load via Chebyshevs

Letting R_i be the number of requests sent to server i, $\mathbb{E}[\mathsf{R}_i] = \frac{n}{k}$ and $\mathrm{Var}[\mathsf{R}_i] \leq \frac{n}{k}$.

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Applying Chebyshev's:

$$\Pr\left(\mathsf{R}_i \geq \frac{2n}{k}\right) \leq \Pr\left(|\mathsf{R}_i - \mathbb{E}[\mathsf{R}_i]| \geq \frac{n}{k}\right) \leq \frac{\mathrm{Var}(R_i)}{(n/k)^2} = \frac{n/k}{(n/k)^2} = \frac{k}{n}$$

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- Overload probability is extremely small when $k \ll n$!

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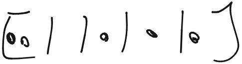
n servers
n requests



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- Overload probability is extremely small when $k \ll n$!
- Might seem counterintuitive – bound gets worse as k grows.
- When k is large, the number of requests each server sees in expectation is very small so the law of large numbers doesn't 'kick in'.

n: total number of requests, k: number of servers randomly assigned requests, R_i: number of requests assigned to server i.

Maximum Server Load

What is the probability that the maximum server load exceeds $2 \cdot \mathbb{E}[R_i] = \frac{2n}{k}$. I.e., that some server is overloaded if we give each $\frac{2n}{k}$ capacity?

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What is the probability that the maximum server load exceeds $2 \cdot \mathbb{E}[\mathrm{R}_i] = \frac{2n}{k}$. I.e., that some server is overloaded if we give each $\frac{2n}{k}$ capacity?

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$$\Pr\left(\max_i(\mathsf{R}_i) \geq \frac{2n}{k}\right) = \Pr\left(\left[\mathsf{R}_1 \geq \frac{2n}{k}\right] \cup \left[\mathsf{R}_2 \geq \frac{2n}{k}\right] \cup \ldots \cup \left[\mathsf{R}_k \geq \frac{2n}{k}\right]\right)$$

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$$\Pr\left(\max_i(R_i) \geq \frac{2n}{k}\right) = \Pr\left(\left[R_1 \geq \frac{2n}{k}\right] \text{ or } \left[R_2 \geq \frac{2n}{k}\right] \text{ or } \ldots \text{ or } \left[R_k \geq \frac{2n}{k}\right]\right)$$

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We want to show that $\Pr\left(\bigcup_{i=1}^{k}\left[R_i \geq \frac{2n}{k}\right]\right)$ is small.

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We want to show that $\Pr\left(\bigcup_{i=1}^{k}\left[R_i \geq \frac{2n}{k}\right]\right)$ is small.

How do we do this? Note that $R_1, \ldots, R_k$ are correlated in a somewhat complex way.

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The Union Bound

Union Bound: For any random events $A_1, A_2, ..., A_k$,

$$\Pr(A_1 \cup A_2 \cup \ldots \cup A_k) \leq \Pr(A_1) + \Pr(A_2) + \ldots + Pr(A_k).$$

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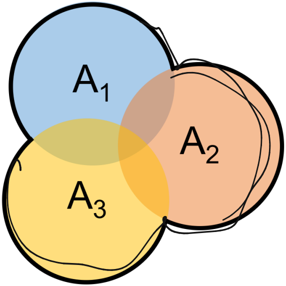

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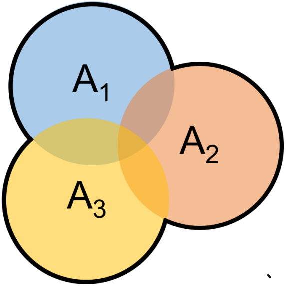

independent

disjoint

When is the union bound tight?

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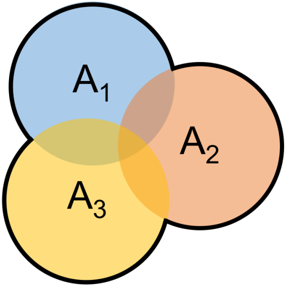

When is the union bound tight? When $A_1, ..., A_k$ are all disjoint.

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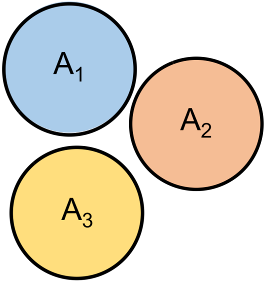

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As long as $k \leq O(\sqrt{n})$, with good probability, the maximum server load will be small (compared to the expected load).

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