# COMPSCI 514: Algorithms for Data Science

Cameron Musco

University of Massachusetts Amherst. Fall 2022.

Lecture 18

# Logistics

- Problem Set 3 is due Monday at 11:59pm.
- No quiz due Monday.
- I will hold additional office hours on Thursday from 11:30am-12:40pm.

# Summary

**Last Class**

- The Singular Value Decomposition (SVD) and its connection to eigendecomposition of $X^T X$ and $XX^T$, and low-rank approximation.

**This Class: Application of Low-Rank Approximation Beyond Compression**

- Low-rank matrix completion (predicting missing measurements using low-rank structure).
- Entity embeddings (e.g., word embeddings, node embeddings).
- Low-rank approximation for non-linear dimensionality reduction.
- Eigendecomposition to partition graphs into clusters.

# SVD Review

- Every $\mathbf{X} \in \mathbb{R}^{n \times d}$ can be written in its SVD as $\mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$.
- $\mathbf{U} \in \mathbb{R}^{n \times r}$ (orthonormal) contains the eigenvectors of $\mathbf{X}\mathbf{X}^T$.
  $\mathbf{V} \in \mathbb{R}^{d \times r}$ (orthonormal) contains the eigenvectors of $\mathbf{X}^\mathsf{T}\mathbf{X}$.
  $\mathbf{\Sigma} \in \mathbb{R}^{r \times r}$ (diagonal) contains their eigenvalues.
- $\mathbf{U}_k\mathbf{U}_k^T\mathbf{X} = \mathbf{X}\mathbf{V}_k\mathbf{V}_k^T = \mathbf{U}_k\mathbf{\Sigma}_k\mathbf{V}_k^T = \underset{\mathbf{B} \text{ s.t. } \operatorname{rank}(\mathbf{B}) \leq k}{\arg\min} \|\mathbf{X} - \mathbf{B}\|_F$.

# Matrix Completion

Consider a matrix $\mathbf{X} \in \mathbb{R}^{n \times d}$ which we cannot fully observe but believe is close to rank-$k$ (i.e., well approximated by a rank $k$ matrix). Classic example: the Netflix prize problem.

**X** Movies

Users

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 5 | 3 | 3 | 1 | 4 | 4 | 4 | 3 | 5 |
| 4 | 3 | 3 | 1 | 4 | 4 | 5 | 3 | 5 |
| 3 | 3 | 3 | 2 | 3 | 3 | 3 | 3 | 3 |
| 4 | 3 | 3 | 4 | 4 | 4 | 4 | 3 | 3 |
| 3 | 3 | 3 | 2 | 3 | 3 | 3 | 3 | 3 |
| 2 | 5 | 3 | 4 | 4 | 4 | 4 | 4 | 5 |
| 1 | 3 | 3 | 2 | 3 | 3 | 3 | 1 | 2 |

Solve: $$Y = \underset{\mathbf{B} \text{ s.t. } \operatorname{rank}(\mathbf{B}) \leq k}{\arg\min} \sum_{\text{observed } (j,k)} \left[\mathbf{X}_{j,k} - \mathbf{B}_{j,k}\right]^2$$

Under certain assumptions, can show that $\mathbf{Y}$ well approximates $\mathbf{X}$ on both the observed and (most importantly) unobserved entries.

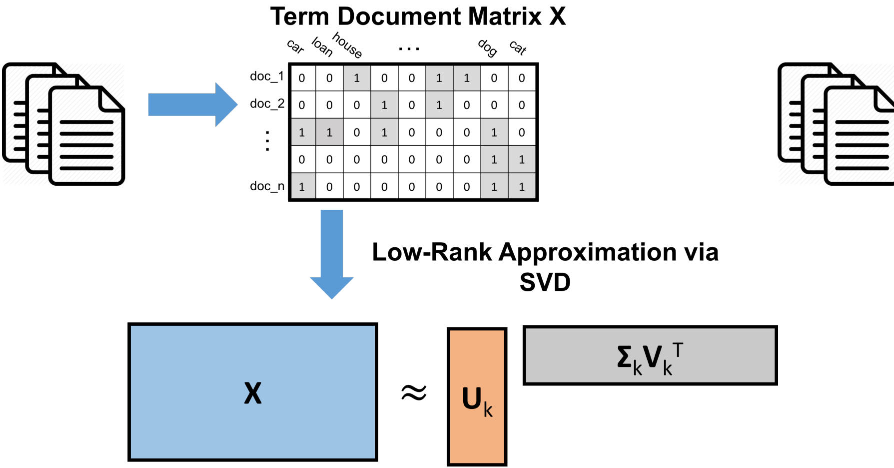
# Entity Embeddings

Dimensionality reduction embeds $d$-dimensional vectors into $k$ dimensions. But what about when you want to embed objects other than vectors?

- Documents (for topic-based search and classification)
- Words (to identify synonyms, translations, etc.)
- Nodes in a social network

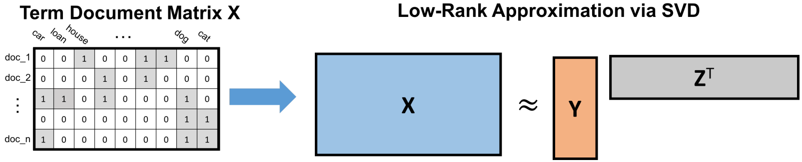
**Classic Approach:** Convert each item into a (very) high-dimensional feature vector and then apply low-rank approximation.

# Example: Latent Semantic Analysis




| | car | loan | house | … | | | | dog | cat |
|---|---|---|---|---|---|---|---|---|---|
| doc_1 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 |
| doc_2 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 |
| ⋮ | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 |
| | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| doc_n | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |

# Example: Latent Semantic Analysis



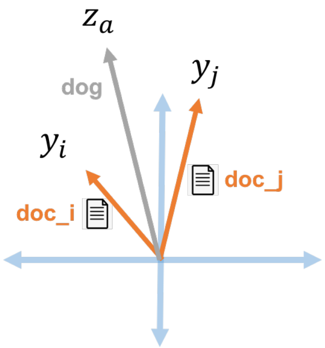

- If the error $\|X - YZ^T\|_F$ is small, then on average,

$$X_{i,a} \approx (YZ^T)_{i,a} = \langle \vec{y}_i, \vec{z}_a \rangle.$$

- I.e., $\langle \vec{y}_i, \vec{z}_a \rangle \approx 1$ when $doc_i$ contains $word_a$.
- If $doc_i$ and $doc_j$ both contain $word_a$, $\langle \vec{y}_i, \vec{z}_a \rangle \approx \langle \vec{y}_j, \vec{z}_a \rangle \approx 1$.

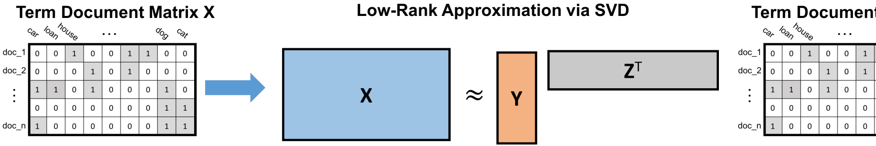
# Example: Latent Semantic Analysis

If $doc_i$ and $doc_j$ both contain $word_a$, $\langle \vec{y}_i, \vec{z}_a \rangle \approx \langle \vec{y}_j, \vec{z}_a \rangle \approx 1$




**Another View:** Each column of **Y** represents a 'topic'. $\vec{y}_i(j)$ indicates how much $doc_i$ belongs to topic $j$. $\vec{z}_a(j)$ indicates how much $word_a$ associates with that topic.

# Example: Latent Semantic Analysis



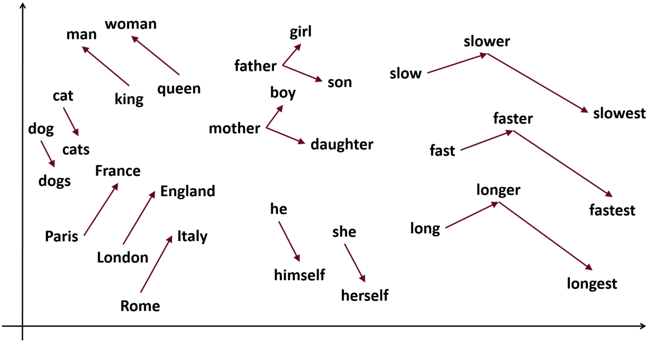

- Just like with documents, $\vec{z}_a$ and $\vec{z}_b$ will tend to have high dot product if $word_a$ and $word_b$ appear in many of the same documents.
- In an SVD decomposition we set $\mathbf{Z}^T = \mathbf{\Sigma}_k \mathbf{V}_K^T$.
- The columns of $\mathbf{V}_k$ are equivalently: the top $k$ eigenvectors of $\mathbf{X}^T\mathbf{X}$.
- **Claim:** $\mathbf{Z}\mathbf{Z}^\mathsf{T}$ is the best rank-$k$ approximation of $\mathbf{X}^T\mathbf{X}$. I.e., $\arg\min_{\text{rank}-k\ \mathbf{B}} \|\mathbf{X}^T\mathbf{X} - \mathbf{B}\|_F$

# Example: Word Embedding

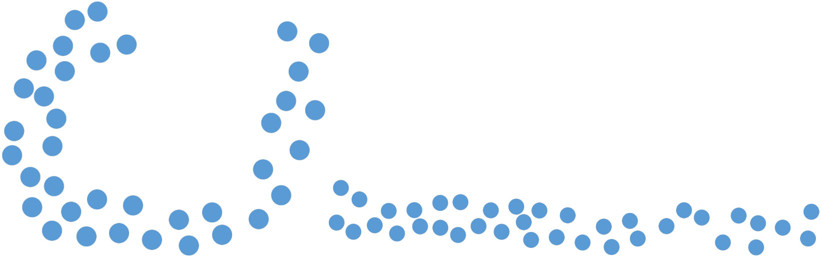
LSA gives a way of embedding words into $k$-dimensional space.

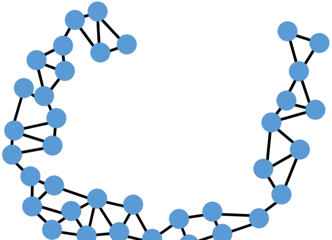
- Embedding is via low-rank approximation of $\mathbf{X}^T\mathbf{X}$: where $(\mathbf{X}^T\mathbf{X})_{a,b}$ is the number of documents that both $word_a$ and $word_b$ appear in.
- Think about $\mathbf{X}^T\mathbf{X}$ as a similarity matrix (gram matrix, kernel matrix) with entry $(a, b)$ being the similarity between $word_a$ and $word_b$.
- Many ways to measure similarity: number of sentences both occur in, number of times both appear in the same window of $w$ words, in similar positions of documents in different languages, etc.
- Replacing $\mathbf{X}^T\mathbf{X}$ with these different metrics (sometimes appropriately transformed) leads to popular word embedding algorithms: word2vec, GloVe, fastText, etc.

# Example: Word Embedding




**Note:** word2vec is typically described as a neural-network method, but can be viewed as just a low-rank approximation of a specific similarity matrix. *Neural word embedding as implicit matrix factorization*, Levy and Goldberg.

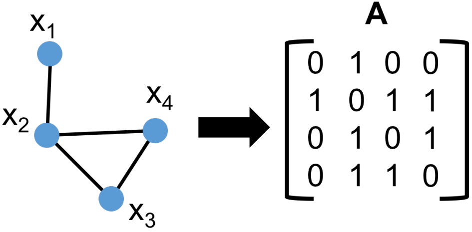




# Linear Algebraic Representation of a Graph

Once we have connected $n$ data points $x_1, \ldots, x_n$ into a graph, we can represent that graph by its (weighted) adjacency matrix.

$$\mathbf{A} \in \mathbb{R}^{n \times n} \text{ with } \mathbf{A}_{i,j} = \text{ edge weight between nodes } i \text{ and } j$$



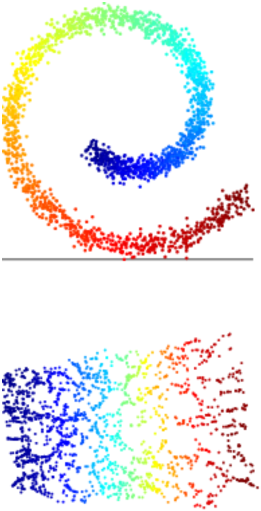

In LSA example, when $\mathbf{X}$ is the term-document matrix, $\mathbf{X}^T\mathbf{X}$ is like an adjacency matrix, where $word_a$ and $word_b$ are connected if they appear in at least 1 document together (edge weight is # documents they appear in together).

# Adjacency Matrix Eigenvectors

How do we compute an optimal low-rank approximation of $\mathbf{A}$?

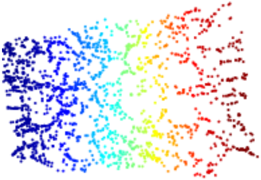
- Project onto the top $k$ eigenvectors of $\mathbf{A}^T\mathbf{A} = \mathbf{A}^2$. These are just the eigenvectors of $\mathbf{A}$.
- $\mathbf{A} \approx \mathbf{A}\mathbf{V}\mathbf{V}^T$. The rows of $\mathbf{A}\mathbf{V}$ can be thought of as 'embeddings' for the vertices.
- Similar vertices (close with regards to graph proximity) should have similar embeddings.

# Spectral Embedding



**Step 1:** Produce a nearest neighbor graph based on your input data in $\mathbb{R}^d$.
**Step 2:** Apply low-rank approximation to the graph adjacency matrix to produce embeddings in $\mathbb{R}^k$.
**Step 3:** Work with the data in the embedded space. Where distances represent distances in your original 'non-linear space.'

# Spectral Embedding



What other methods do you know for embedding or representing data points with non-linear structure?

Questions?