COMPSCI 514: Algorithms for Data Science

Cameron Musco

University of Massachusetts Amherst. Fall 2022.

Lecture 12

Logistics

- Problem Set 2 is due Friday, 11:59pm.
- No quiz this week.
- The exam will be held next Thursday in class.
- We will do some midterm review in class on Tuesday. I will also hold additional office hours for midterm prep, TBD.

Summary

Last Class: The Johnson-Lindenstrauss Lemma

- Intro to dimensionality reduction and low-distortion embeddings.
- Statement of the JL Lemma: we can obtain low-distortion embeddings for any set of points via random projection.

This Class:

- Reduction of the JL Lemma to the ‘distributional JL Lemma’.
- Proof of the distributional JL lemma.
- Example application to clustering.

The Johnson-Lindenstrauss Lemma

Johnson-Lindenstrauss Lemma: For any set of points $\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d$ and $\epsilon > 0$ there exists a linear map $\mathbf{\Pi} : \mathbb{R}^d \to \mathbb{R}^m$ such that $m = O\left(\frac{\log n}{\epsilon^2}\right)$ and letting $\tilde{x}_i = \mathbf{\Pi}\vec{x}_i$:

$$\text{For all } i, j: \; (1-\epsilon)\|\vec{x}_i - \vec{x}_j\|_2 \leq \|\tilde{x}_i - \tilde{x}_j\|_2 \leq (1+\epsilon)\|\vec{x}_i - \vec{x}_j\|_2.$$

Further, if $\mathbf{\Pi} \in \mathbb{R}^{m \times d}$ has each entry chosen i.i.d. from $\mathcal{N}(0, 1/m)$, it satisfies the guarantee with high probability.

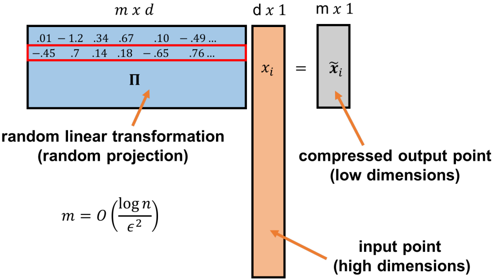
For $d = 1$ trillion, $\epsilon = .05$, and $n = 100,000$, $m \approx 6600$.

Very surprising! Powerful result with a simple construction: applying a random linear transformation to a set of points preserves distances between all those points with high probability.

Random Projection

For any $\vec{x}_1, \ldots, \vec{x}_n$ and $\mathbf{\Pi} \in \mathbb{R}^{m \times d}$ with each entry chosen i.i.d. from $\mathcal{N}(0, 1/m)$, with high probability, letting $\tilde{\mathbf{x}}_i = \mathbf{\Pi}\vec{x}_i$:

$$\text{For all } i, j: \; (1-\epsilon)\|\vec{x}_i - \vec{x}_j\|_2 \leq \|\tilde{\mathbf{x}}_i - \tilde{\mathbf{x}}_j\|_2 \leq (1+\epsilon)\|\vec{x}_i - \vec{x}_j\|_2.$$



- $\mathbf{\Pi}$ is known as a random projection. It is a random linear function, mapping length d vectors to length m vectors.
- $\mathbf{\Pi}$ is data oblivious. Stark contrast to methods like PCA.

Algorithmic Considerations

- Many alternative constructions: ± 1 entries, sparse (most entries 0), Fourier structured, etc. $\implies$ more efficient computation of $\tilde{\mathbf{x}}_i = \mathbf{\Pi}\vec{x}_i$.
- Data oblivious property means that once $\mathbf{\Pi}$ is chosen, $\tilde{\mathbf{x}}_1, \ldots, \tilde{\mathbf{x}}_n$ can be computed in a stream with little memory.
- Memory needed is just $O(d + nm)$ vs. $O(nd)$ to store the full data set.
- Compression can also be easily performed in parallel on different servers.
- When new data points are added, can be easily compressed, without updating existing points.

Distributional JL

The Johnson-Lindenstrauss Lemma is a direct consequence of a closely related lemma:

Distributional JL Lemma: Let $\mathbf{\Pi} \in \mathbb{R}^{m \times d}$ have each entry chosen i.i.d. as $\mathcal{N}(0, 1/m)$. If we set $m = O\left(\frac{\log(1/\delta)}{\epsilon^2}\right)$, then for any $\vec{y} \in \mathbb{R}^d$, with probability $\geq 1 - \delta$

$$(1-\epsilon)\|\vec{y}\|_2 \leq \|\mathbf{\Pi}\vec{y}\|_2 \leq (1+\epsilon)\|\vec{y}\|_2$$

Applying a random matrix $\mathbf{\Pi}$ to any vector $\vec{y}$ preserves $\vec{y}$'s norm with high probability.

- Like a low-distortion embedding, but for the length of a compressed vector rather than distances between vectors.
- Can be proven from first principles.

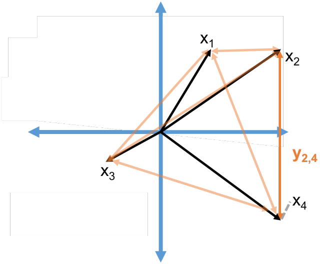
$\mathbf{\Pi} \in \mathbb{R}^{m \times d}$: random projection matrix. d: original dimension. m: compressed dimension, ϵ: embedding error, δ: embedding failure prob.

Distributional JL $\implies$ JL

Distributional JL Lemma $\implies$ JL Lemma: Distributional JL show that a random projection $\mathbf{\Pi}$ preserves the norm of any y. The main JL Lemma says that $\mathbf{\Pi}$ preserves distances between vectors.

Since $\mathbf{\Pi}$ is linear these are the same thing!

Proof: Given $\vec{x}_1, \ldots, \vec{x}_n$, define $\binom{n}{2}$ vectors $\vec{y}_{ij}$ where $\vec{y}_{ij} = \vec{x}_i - \vec{x}_j$.



- If we choose $\mathbf{\Pi}$ with $m = O\left(\frac{\log 1/\delta}{\epsilon^2}\right)$, for each $\vec{y}_{ij}$ with probability $\geq 1 - \delta$ we have:

$$(1-\epsilon)\|\vec{y}_{ij}\vec{x}_i - \vec{x}_j\|_2 \leq \|\mathbf{\Pi}\vec{y}_{ij}\mathbf{\Pi}(\vec{x}_i - \vec{x}_j)\tilde{\mathbf{x}}_i - \tilde{\mathbf{x}}_j\|_2 \leq (1+\epsilon)\|\vec{y}_{ij}\vec{x}_i - \vec{x}_j\|_2$$

Claim: If we choose $\mathbf{\Pi}$ with i.i.d. $\mathcal{N}(0, 1/m)$ entries and $m = O\left(\frac{\log(1/\delta')}{\epsilon^2}\right)$, letting $\tilde{\mathbf{x}}_i = \mathbf{\Pi}\vec{x}_i$, for each pair $\vec{x}_i, \vec{x}_j$ with probability $\geq 1 - \delta'$ we have:

$$(1-\epsilon)\|\vec{x}_i - \vec{x}_j\|_2 \leq \|\tilde{\mathbf{x}}_i - \tilde{\mathbf{x}}_j\|_2 \leq (1+\epsilon)\|\vec{x}_i - \vec{x}_j\|_2.$$

With what probability are all pairwise distances preserved?

Union bound: With probability $\geq 1 - \binom{n}{2} \cdot \delta'$ all pairwise distances are preserved.

Apply the claim with $\delta' = \delta / \binom{n}{2}$. $\implies$ for $m = O\left(\frac{\log(1/\delta')}{\epsilon^2}\right)$, all pairwise distances are preserved with probability $\geq 1 - \delta$.

$$m = O\left(\frac{\log(1/\delta')}{\epsilon^2}\right) = O\left(\frac{\log(\binom{n}{2}/\delta)}{\epsilon^2}\right) = O\left(\frac{\log(n^2/\delta)}{\epsilon^2}\right) = O\left(\frac{\log(n/\delta)}{\epsilon^2}\right)$$

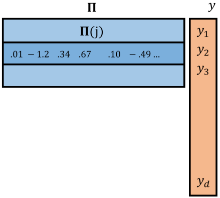
Yields the JL lemma.

Distributional JL Proof

Distributional JL Lemma: Let $\mathbf{\Pi} \in \mathbb{R}^{m \times d}$ have each entry chosen i.i.d. as $\mathcal{N}(0, 1/m)$. If we set $m = O\left(\frac{\log(1/\delta)}{\epsilon^2}\right)$, then for any $\vec{y} \in \mathbb{R}^d$, with probability $\geq 1 - \delta$

$$(1-\epsilon)\|\vec{y}\|_2 \leq \|\mathbf{\Pi}\vec{y}\|_2 \leq (1+\epsilon)\|\vec{y}\|_2$$

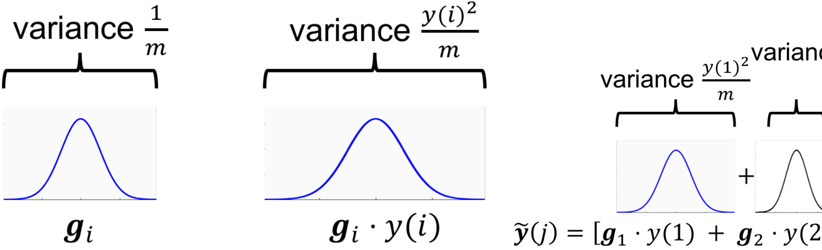
- Let $\tilde{\mathbf{y}}$ denote $\mathbf{\Pi}\vec{y}$ and let $\mathbf{\Pi}(j)$ denote the j^{th} row of $\mathbf{\Pi}$.
- For any j, $\tilde{\mathbf{y}}(j) = \langle \mathbf{\Pi}(j), \vec{y} \rangle = \sum_{i=1}^{d} \mathbf{g}_i \cdot \vec{y}(i)$ where $\mathbf{g}_i \sim \mathcal{N}(0, 1/m)$.



$\vec{y} \in \mathbb{R}^d$: arbitrary vector, $\tilde{\mathbf{y}} \in \mathbb{R}^m$: compressed vector, $\mathbf{\Pi} \in \mathbb{R}^{m \times d}$: random projection. d: original dim. m: compressed dim, ϵ: error, δ: failure prob.

Distributional JL Proof

- Let $\tilde{\mathbf{y}}$ denote $\mathbf{\Pi}\vec{y}$ and let $\mathbf{\Pi}(j)$ denote the j^{th} row of $\mathbf{\Pi}$.
- For any j, $\tilde{\mathbf{y}}(j) = \langle \mathbf{\Pi}(j), \vec{y} \rangle = \sum_{i=1}^{d} \mathbf{g}_i \cdot \vec{y}(i)$ where $\mathbf{g}_i \sim \mathcal{N}(0, 1/m)$.
- $\mathbf{g}_i \cdot \vec{y}(i) \sim \mathcal{N}(0, \frac{\vec{y}(i)^2}{m})$: normally distributed with variance $\frac{\vec{y}(i)^2}{m}$.



What is the distribution of $\tilde{\mathbf{y}}(j)$? Also Gaussian!

$\vec{y} \in \mathbb{R}^d$: arbitrary vector, $\tilde{\mathbf{y}} \in \mathbb{R}^m$: compressed vector, $\mathbf{\Pi} \in \mathbb{R}^{m \times d}$: random projection mapping $\vec{y} \to \tilde{\mathbf{y}}$. $\mathbf{\Pi}(j)$: j^{th} row of $\mathbf{\Pi}$, d: original dimension. m: compressed dimension, $\mathbf{g}_i$: normally distributed random variable.

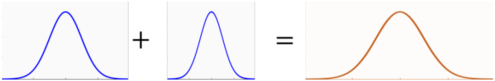
Distributional JL Proof

Letting $\tilde{\mathbf{y}} = \mathbf{\Pi}\vec{y}$, we have $\tilde{\mathbf{y}}(j) = \langle \mathbf{\Pi}(j), \vec{y} \rangle$ and:

$$\tilde{\mathbf{y}}(j) = \sum_{i=1}^{d} \mathbf{g}_i \cdot \vec{y}(i) \text{ where } \mathbf{g}_i \cdot \vec{y}(i) \sim \mathcal{N}\left(0, \frac{\vec{y}(i)^2}{m}\right).$$

Stability of Gaussian Random Variables. For independent $a \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $b \sim \mathcal{N}(\mu_2, \sigma_2^2)$ we have:

$$a + b \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$



Thus, $\tilde{\mathbf{y}}(j) \sim \mathcal{N}(0, \frac{\vec{y}(1)^2}{m} + \frac{\vec{y}(2)^2}{m} + \ldots + \frac{\vec{y}(d)^2}{m}\ \frac{\|\vec{y}\|_2^2}{m})$ I.e., $\tilde{\mathbf{y}}$ itself is a random Gaussian vector. Rotational invariance of the Gaussian distribution.

$\vec{y} \in \mathbb{R}^d$: arbitrary vector, $\tilde{\mathbf{y}} \in \mathbb{R}^m$: compressed vector, $\mathbf{\Pi} \in \mathbb{R}^{m \times d}$: random projection mapping $\vec{y} \to \tilde{\mathbf{y}}$. $\mathbf{\Pi}(j)$: j^{th} row of $\mathbf{\Pi}$, d: original dimension. m: com-

Distributional JL Proof

So far: Letting $\mathbf{\Pi} \in \mathbb{R}^{d \times m}$ have each entry chosen i.i.d. as $\mathcal{N}(0, 1/m)$, for any $\vec{y} \in \mathbb{R}^d$, letting $\tilde{\mathbf{y}} = \mathbf{\Pi}\vec{y}$:

$$\tilde{\mathbf{y}}(j) \sim \mathcal{N}(0, \|\vec{y}\|_2^2/m).$$

What is $\mathbb{E}[\|\tilde{\mathbf{y}}\|_2^2]$?

$$\mathbb{E}[\|\tilde{\mathbf{y}}\|_2^2] = \mathbb{E}\left[\sum_{j=1}^m \tilde{\mathbf{y}}(j)^2\right] = \sum_{j=1}^m \mathbb{E}[\tilde{\mathbf{y}}(j)^2]$$

$$= \sum_{j=1}^m \frac{\|\vec{y}\|_2^2}{m} = \|\vec{y}\|_2^2$$

So $\tilde{\mathbf{y}}$ has the right norm in expectation.

How is $\|\tilde{\mathbf{y}}\|_2^2$ distributed? Does it concentrate?

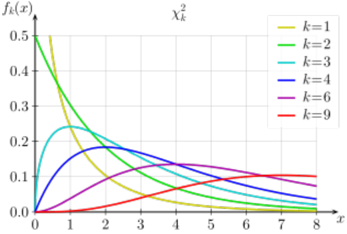
$\vec{y} \in \mathbb{R}^d$: arbitrary vector, $\tilde{\mathbf{y}} \in \mathbb{R}^m$: compressed vector, $\mathbf{\Pi} \in \mathbb{R}^{m \times d}$: random projection mapping $\vec{y} \to \tilde{\mathbf{y}}$. $\mathbf{\Pi}(j)$: j^{th} row of $\mathbf{\Pi}$, d: original dimension. m: compressed dimension, $\mathbf{g}_i$: normally distributed random variable

Distributional JL Proof

So far: Letting $\mathbf{\Pi} \in \mathbb{R}^{d \times m}$ have each entry chosen i.i.d. as $\mathcal{N}(0, 1/m)$, for any $\vec{y} \in \mathbb{R}^d$, letting $\tilde{\mathbf{y}} = \mathbf{\Pi}\vec{y}$:

$$\tilde{\mathbf{y}}(j) \sim \mathcal{N}(0, \|\vec{y}\|_2^2/m) \text{ and } \mathbb{E}[\|\tilde{\mathbf{y}}\|_2^2] = \|\vec{y}\|_2^2$$

$\|\tilde{\mathbf{y}}\|_2^2 = \sum_{i=1}^m \tilde{\mathbf{y}}(j)^2$ a Chi-Squared random variable with m degrees of freedom (a sum of m squared independent Gaussians)

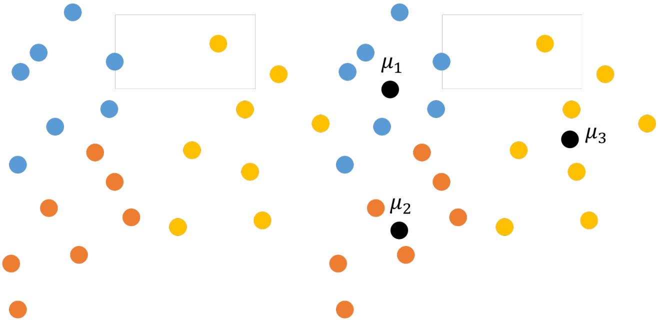


Lemma: (Chi-Squared Concentration) Letting $\mathbf{Z}$ be a Chi-Squared random variable with m degrees of freedom,

$$\Pr\left[|\mathbf{Z} - \mathbb{E}\mathbf{Z}| \geq \epsilon \mathbb{E}\mathbf{Z}\right] \leq 2e^{-m\epsilon^2/8}.$$

Example Application: k-means clustering

Goal: Separate n points in d dimensional space into k groups.



k-means Objective: $Cost(\mathcal{C}_1, \ldots, \mathcal{C}_k) = \min_{\mathcal{C}_1, \ldots \mathcal{C}_k} \sum_{j=1}^{k} \sum_{\vec{x} \in \mathcal{C}_k} \|\vec{x} - \mu_j\|_2^2.$

Write in terms of distances:

$$Cost(\mathcal{C}_1, \ldots, \mathcal{C}_k) = \min_{\mathcal{C}_1, \ldots \mathcal{C}_k} \sum_{j=1}^{k} \sum_{\vec{x}_1, \vec{x}_2 \in \mathcal{C}_k} \|\vec{x}_1 - \vec{x}_2\|_2^2$$

Example Application: k-means clustering

k-means Objective: $Cost(\mathcal{C}_1, \ldots, \mathcal{C}_k) = \min_{\mathcal{C}_1, \ldots \mathcal{C}_k} \sum_{j=1}^{k} \sum_{\vec{x}_1, \vec{x}_2 \in \mathcal{C}_k} \|\vec{x}_1 - \vec{x}_2\|_2^2$

If we randomly project to $m = O\left(\frac{\log n}{\epsilon^2}\right)$ dimensions, for all pairs $\vec{x}_1, \vec{x}_2$,

$$(1-\epsilon)\|\vec{x}_1 - \vec{x}_2\|_2^2 \leq \|\tilde{\mathbf{x}}_1 - \tilde{\mathbf{x}}_2\|_2^2 \leq (1+\epsilon)\|\vec{x}_1 - \vec{x}_2\|_2^2 \implies$$

Letting $\overline{Cost}(\mathcal{C}_1, \ldots, \mathcal{C}_k) = \min_{\mathcal{C}_1, \ldots \mathcal{C}_k} \sum_{j=1}^{k} \sum_{\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2 \in \mathcal{C}_k} \|\tilde{\mathbf{x}}_1 - \tilde{\mathbf{x}}_2\|_2^2$

$$(1-\epsilon)Cost(\mathcal{C}_1, \ldots, \mathcal{C}_k) \leq \overline{Cost}(\mathcal{C}_1, \ldots, \mathcal{C}_k) \leq (1+\epsilon)Cost(\mathcal{C}_1, \ldots, \mathcal{C}_k).$$

Upshot: Can cluster in m dimensional space (much more efficiently) and minimize $\overline{Cost}(\mathcal{C}_1, \ldots, \mathcal{C}_k)$. The optimal set of clusters will have true cost within $1 + c\epsilon$ times the true optimal. Good exercise to prove this.