COMPSCI 514: Algorithms for Data Science

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University of Massachusetts Amherst. Fall 2022.

Lecture 11

Logistics

4.1 tables of size $O(n)$

tables of size $n = O(k)$

- Problem Set 2 is due on Friday at 11:59pm.
- Midterm is in class next Thursday, 10/20.
- I have posted a study guide and practice questions on the course schedule.

$\frac{n_k}{m}$ $\quad n - \sum_{i=1}^{k} f_i$

Summary

Last Class:

- Introduced the k-frequent elements problem – identify all elements of a stream of n elements that occur $\geq n/k$ times.
- Saw how to solve approximately in $O(k \log n/\epsilon)$ space using the Count-min sketch algorithm.
- Simple analysis based on Markov's inequality and repeated random hashing.

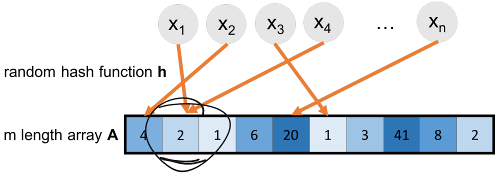
This Class:

- Recap and finish up Count-min sketch
- Randomized methods for dimensionality reduction.
- The Johnson-Lindenstrauss Lemma.

Count-Min Sketch

(ε, k) - Freq. Items Problem

Goal: Return all items in a stream of n elements with frequency at least n/k. Don't return any with frequency $\leq (1-\epsilon)\cdot\frac{n}{k}$.



$F(x_1)$

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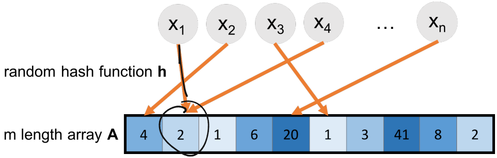

- $A[\mathbf{h}(x)] = f(x) + \sum_{y\neq x:\mathbf{h}(y)=\mathbf{h}(x)} f(y)$ where $\mathbb{E}[\sum_{y\neq x:\mathbf{h}(y)=\mathbf{h}(x)} f(y)] \leq \frac{n}{m}$.

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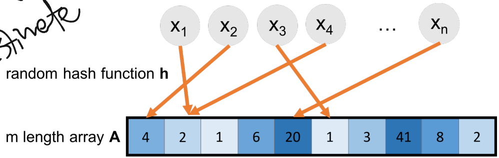
$f(x)$ = true freq. of x

$\tilde{f}(x)$ is an estimate



- $A[\mathbf{h}(x)] = f(x) + \sum_{y \neq x : \mathbf{h}(y) = \mathbf{h}(x)} f(y)$ where $\mathbb{E}[\sum_{y \neq x : \mathbf{h}(y) = \mathbf{h}(x)} f(y)] \leq \frac{n}{m}$.

  $\Pr\left(\tilde{f}(x) \geq f(x) + \frac{2n}{m}\right) \leq \frac{1}{2}$

- If we let $\tilde{f}(x)$ be the minimum of $t = \log_2(1/\delta)$ estimates, $f(x) \leq \tilde{f}(x) \leq f(x) + \frac{2n}{m}$ with probability at least $1 - \delta$.

  $1 - \frac{1}{2^t} = 1 - \delta$

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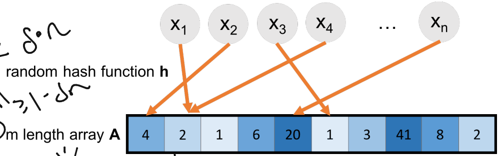

- $A[\mathbf{h}(x)] = f(x) + \sum_{y \neq x : \mathbf{h}(y)=\mathbf{h}(x)} f(y)$ where $\mathbb{E}[\sum_{y \neq x : \mathbf{h}(y)=\mathbf{h}(x)} f(y)] \leq \frac{n}{m}$.
- If we let $\tilde{f}(x)$ be the minimum of $t = \log_2(1/\delta)$ estimates, $f(x) \leq \tilde{f}(x) \leq f(x) + \frac{2n}{m}$ with probability at least $1-\delta$.
- Setting $m = O(k/\epsilon)$, $\delta = O(\delta'/n)$, and applying a union bound, we have a good estimate for all $f(x)$ with probability at least $1-\delta'$.

Identifying Frequent Elements

Count-min sketch gives an accurate frequency estimate for every item in the stream. But how do we identify the frequent items without having to store/look up the estimated frequency for all elements in the stream?

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One approach:

- When a new item comes in at step i, check if its estimated frequency is $\geq i/k$ and store it if so.
- At step i remove any stored items whose estimated frequency drops below i/k.
- Store at most $O(k)$ items at once and have all items with frequency $\geq n/k$ stored at the end of the stream, no items with frequency $< (1-\epsilon) \cdot \frac{n}{k}$.

High Dimensional Data

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- The human genome contains 3 billion+ base pairs. Genetic datasets often contain information on 100s of thousands+ mutations and genetic markers.

Data as Vectors and Matrices

In data analysis and machine learning, data points with many attributes are often stored, processed, and interpreted as high dimensional vectors, with real valued entries.

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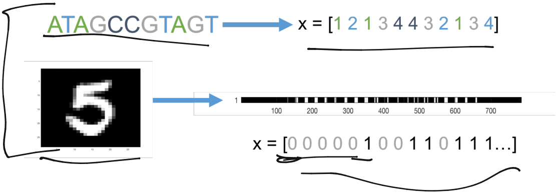

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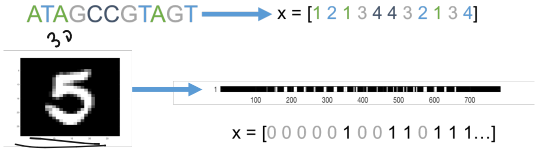

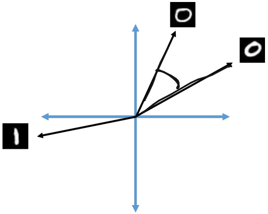

Similarities/distances between vectors (e.g., $\langle x, y \rangle$, $\|x - y\|_2$) have meaning for underlying data points.

Datasets as Vectors and Matrices

Data points are interpreted as high dimensional vectors, with real valued entries. Data set is interpreted as a matrix.

Data Points: $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n \in \mathbb{R}^d$.

Data Set: $X \in \mathbb{R}^{n \times d}$ with i^{th} row equal to $\vec{x}_i$.

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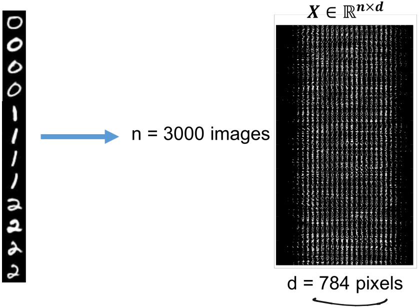

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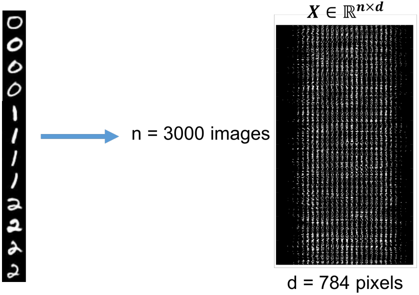

Many data points $n \implies$ tall. Many dimensions $d \implies$ wide.

Dimensionality Reduction

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 → x = [0 0 0 0 0 1 0 0 1 1 0 1 1 1...] → $\tilde{x}$ = [-5.5 4 3.2 -1]

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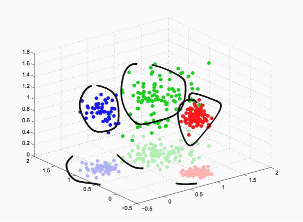
'Lossy compression' that still preserves important information about the relationships between $\vec{x}_1, \ldots, \vec{x}_n$.



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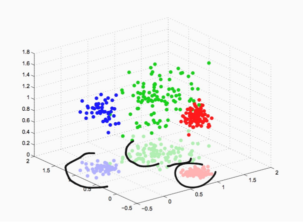

Generally will not consider directly how well $\tilde{x}_i$ approximates $\vec{x}_i$.

Dimensionality Reduction

Dimensionality reduction is one of the most important techniques in data science. What methods have you heard of?

PCA → LSA

T-SNE

Auto encoders, Nueral networks

self-organizing maps

clustering

JPEG, mp3

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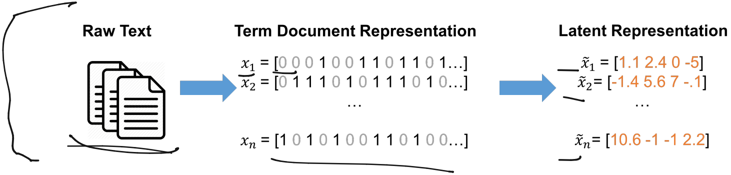

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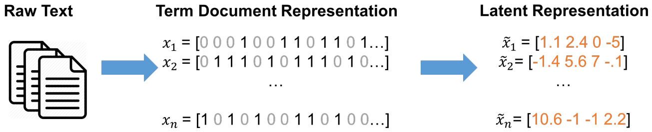

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Compressing data makes it more efficient to work with. May also remove extraneous information/noise.

Embeddings for Euclidean Space

Euclidean Low Distortion Embedding: Given $\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d$ and error parameter $\epsilon \geq 0$, find $\tilde{x}_1, \ldots, \tilde{x}_n \in \mathbb{R}^m$ (where $m \ll d$) such that for all $i, j \in [n]$:

$$(1-\epsilon)\|\vec{x}_i - \vec{x}_j\|_2 \leq \|\tilde{x}_i - \tilde{x}_j\|_2 \leq (1+\epsilon)\|\vec{x}_i - \vec{x}_j\|_2.$$

m-dimension   d-dimension

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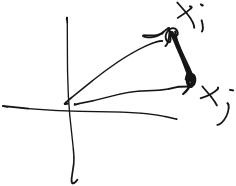
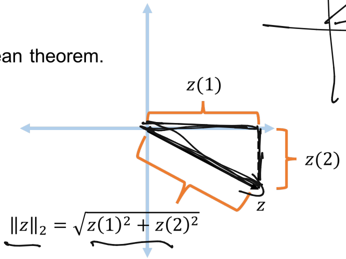
Recall that for $\vec{z} \in \mathbb{R}^n$, $\|\vec{z}\|_2 = \sqrt{\sum_{i=1}^n \vec{z}(i)^2}$.



Pythagorean theorem.



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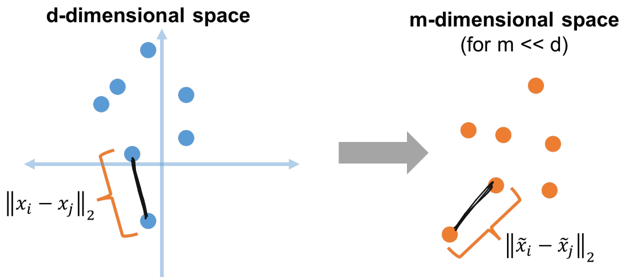

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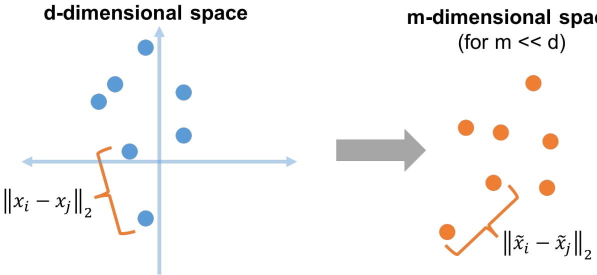
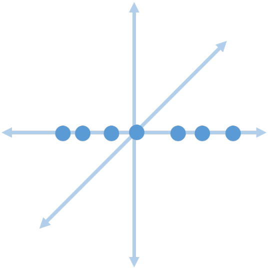


Can use $\tilde{x}_1, \ldots, \tilde{x}_n$ in place of $\vec{x}_1, \ldots, \vec{x}_n$ in clustering, SVM, linear classification, near neighbor search, etc.

Embedding with Assumptions

A very easy case: Assume that $\vec{x}_1, \ldots, \vec{x}_n$ all lie on the 1^{st} axis in $\mathbb{R}^d$.



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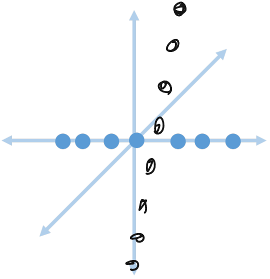

Set $m = 1$ and $\tilde{x}_i = [\vec{x}_i(1)]$ (i.e., $\tilde{x}_i$ contains just a single number).

- $\|\tilde{x}_i - \tilde{x}_j\|_2 = \sqrt{[\vec{x}_i(1) - \vec{x}_j(1)]^2} = |\vec{x}_i(1) - \vec{x}_j(1)| = \|\vec{x}_i - \vec{x}_j\|_2$.

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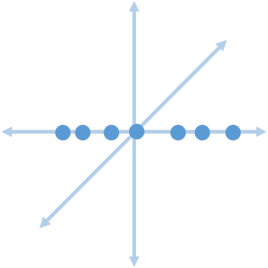

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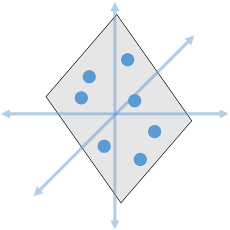
- An embedding with no distortion from any d into $m = 1$.

Embedding with Assumptions

Assume that $\vec{x}_1, \ldots \vec{x}_n$ lie in any k-dimensional subspace $\mathcal{V}$ of $\mathbb{R}^d$.



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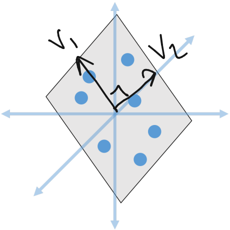

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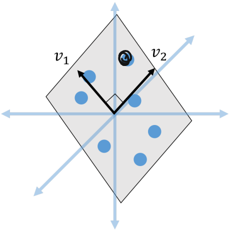
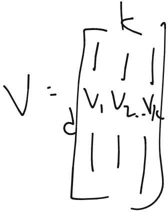


$c_1 \vec{v}_1 + c_2 \vec{v}_2$



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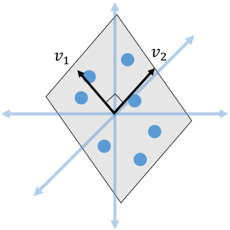

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- For all i, j we have $\vec{x}_i - \vec{x}_j \in \mathcal{V}$ and (a good exercise!):

$$\|\vec{x}_i - \vec{x}_j\|_2 = \sqrt{\sum_{\ell=1}^{k} \langle v_\ell, \vec{x}_i - \vec{x}_j \rangle^2} \qquad .$$

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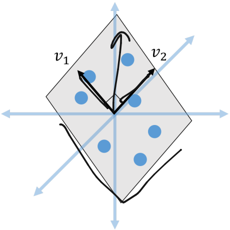

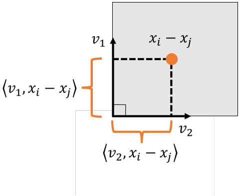

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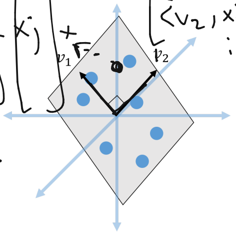

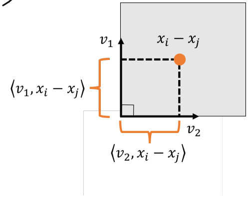

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$$\langle v_\ell, x_i - x_j \rangle$$

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- An embedding with no distortion from any d into $m = k$.
- $\mathbf{V}^T : \mathbb{R}^d \to \mathbb{R}^k$ is a linear map giving our embedding.

Embedding with No Assumptions

What about when we don't make any assumptions on $\vec{x}_1, \ldots, \vec{x}_n$. I.e., they can be scattered arbitrarily around d-dimensional space?

- Can we find a no-distortion embedding into $m \ll d$ dimensions?

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- Can we find a no-distortion embedding into $m \ll d$ dimensions? No. Require $m = d$.
- Can we find an ϵ-distortion embedding into $m \ll d$ dimensions for $\epsilon > 0$? Yes! Always, with m depending on ϵ.

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Johnson-Lindenstrauss Lemma: For any set of points $\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d$ and $\epsilon > 0$ there exists a linear map $\mathbf{\Pi} : \mathbb{R}^d \to \mathbb{R}^m$ such that $m = O\left(\frac{\log n}{\epsilon^2}\right)$ and letting $\tilde{x}_i = \mathbf{\Pi}\vec{x}_i$:

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For $d = 1$ trillion, $\epsilon = .05$, and $n = 100,000$, $m \approx 6600$.

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Very surprising! Powerful result with a simple construction: applying a random linear transformation to a set of points preserves distances between all those points with high probability.

Random Projection

For any $\vec{x}_1, \ldots, \vec{x}_n$ and $\mathbf{\Pi} \in \mathbb{R}^{m \times d}$ with each entry chosen i.i.d. from $\mathcal{N}(0, 1/m)$, with high probability, letting $\tilde{\mathbf{x}}_i = \mathbf{\Pi}\vec{x}_i$:

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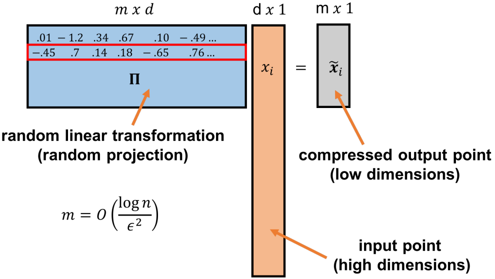

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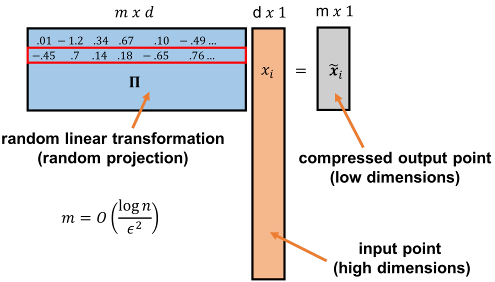

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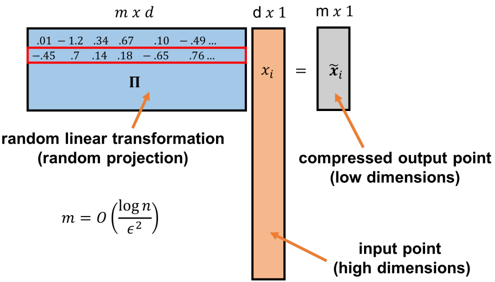

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- $\mathbf{\Pi}$ is data oblivious. Stark contrast to methods like PCA.