COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

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University of Massachusetts Amherst. Fall 2021.

Lecture 4

- Problem set one posted this morning. Due next Friday 9/24, at 8pm.
- Class pacing: Just right – 53/87, A bit too fast – 29/87.
- If things feel way too fast, come to my office hours or send me an email and we can discuss how to make things more manageable.
- **Reminder:** My office hours are now Thursdays 9am on Zoom. Pratheba's in person office hours are Mondays 12pm-1pm in CS 207.

reminder: correct quiz points

Last Class:

- 2-Level Hashing Analysis (linearity of expectation and Markov's inequality)
- 2-universal and pairwise independent hash functions
- Start on hashing for load balancing.

2-universal hash functions are:

✓ a) more efficient to compute than fully independent hash functions.

✓ b) low collision prob.

~~c) are deterministic~~

~~d) zero collisions~~

h(x) = ax+b mod...

h(x)

Last Class:

- 2-Level Hashing Analysis (linearity of expectation and Markov's inequality)
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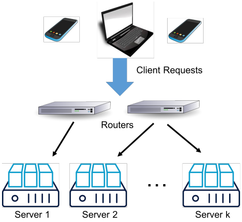
- Start on hashing for load balancing.

This Time:

- Finish hashing for load balancing. Motivating:
  - Stronger concentration inequalities: Chebyshev's inequality, exponential tail bounds, and their connections to the law of large numbers and central limit theorem.
  - The union bound to bound the probability that one of multiple possible correlated events happens.

Randomized Load Balancing:



- n requests randomly assigned to k servers.
- Expected load on server i is $\mathbb{E}[R_i] = \quad$.

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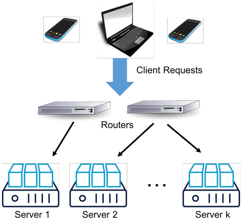

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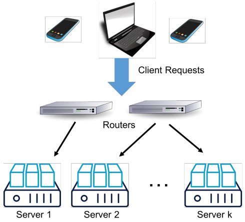

- n requests randomly assigned to k servers.
- Expected load on server i is $\mathbb{E}[\mathbf{R}_i] = \frac{n}{k}$.
- By Markov's inequality, if we provision each server to handle twice this expected load (so $\frac{2n}{k}$ requests), it will be overloaded with probability $\leq 1/2$.

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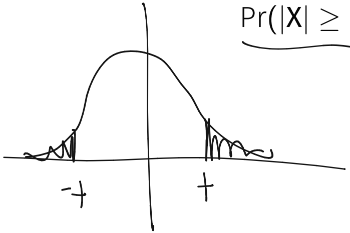
$$\Pr(|X| \geq t) = \Pr(X^2 \geq t^2).$$



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$|X - \mathbb{E}X|$

$$\Pr(|X| \geq t) = \Pr(X^2 \geq t^2) \leq \frac{\mathbb{E}[X^2]}{t^2}.$$

$$\Pr(|X - \mathbb{E}X| \geq t) \leq \frac{\mathbb{E}[(X - \mathbb{E}X)^2]}{t^2} = \frac{\operatorname{Var}(X)}{t^2}$$

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Chebyshev's inequality:

$$\Pr(|X - \mathbb{E}[X]| \geq t) \leq \frac{\mathrm{Var}[X]}{t^2}.$$

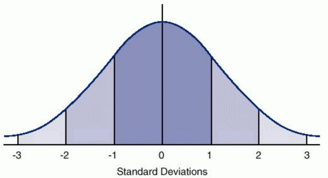
(by plugging in the random variable $X - \mathbb{E}[X]$)

$$\Pr(|X - \mathbb{E}[X]| \geq t) \leq \frac{\mathrm{Var}[X]}{t^2}$$

X: any random variable, t, s: any fixed numbers.

$$\Pr(|X - \mathbb{E}[X]| \geq t) \leq \frac{\text{Var}[X]}{t^2}$$

What is the probability that X falls s standard deviations from it's mean?



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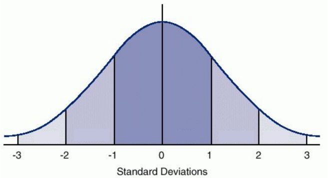

$$\Pr(|X - \mathbb{E}[X]| \geq s \cdot \sqrt{\mathrm{Var}[X]}) \leq \frac{\mathrm{Var}[X]}{s^2 \cdot \mathrm{Var}[X]} = \frac{1}{s^2}.$$

t

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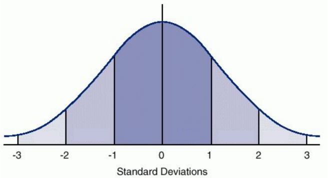

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Why is this so powerful?

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$$\mathbb{E}(S) = \frac{1}{n} \cdot \sum \mathbb{E} X_i = \frac{1}{n} \cdot n \cdot \mu = \mu$$

$$\mathrm{Var}[S] = \mathrm{Var}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right]$$

$$= \frac{1}{n^2} \mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2} \sum \mathrm{Var}(X_i)$$

$$= \frac{1}{n^2} \cdot n \cdot \sigma^2 = \frac{\sigma^2}{n}$$

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By Chebyshev's Inequality: for any fixed value $\epsilon > 0$,

$$\Pr(|S - \mathbb{E}[S]| \geq \epsilon) \leq \frac{\mathrm{Var}[S]}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2}.$$

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LAW OF LARGE NUMBERS

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Law of Large Numbers: with enough samples n, the sample average will always concentrate to the mean.

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- Cannot show from vanilla Markov's inequality.

LOAD BALANCING VARIANCE

We can write the number of requests assigned to server i, R_i as:

$$\mathsf{R}_i = \sum_{j=1}^{n} \mathsf{R}_{i,j}$$

where $\mathsf{R}_{i,j}$ is 1 if request j is assigned to server i and 0 otherwise.

n: total number of requests, k: number of servers randomly assigned requests, R_i: number of requests assigned to server i.

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$$\begin{aligned}\text{Var}[R_{i,j}] &= \mathbb{E}\left[\left(R_{i,j} - \mathbb{E}[R_{i,j}]\right)^2\right] \\ &= \Pr(R_{i,j} = 1) \cdot \left(1 - \mathbb{E}[R_{i,j}]\right)^2 + \Pr(R_{i,j} = 0) \cdot \left(0 - \mathbb{E}[R_{i,j}]\right)^2\end{aligned}$$

$$\frac{1}{k} \cdot \left(1 - \frac{1}{k}\right)^2 + \left(1 - \frac{1}{k}\right) + \left(0 - \frac{1}{k}\right)^2$$

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&= \frac{1}{k} - \frac{1}{k^2} \leq \frac{1}{k}
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&= \frac{1}{k} \cdot \left(1 - \frac{1}{k}\right)^2 + \left(1 - \frac{1}{k}\right) \cdot \left(0 - \frac{1}{k}\right)^2 \\
&= \frac{1}{k} - \frac{1}{k^2} \leq \frac{1}{k} \implies \text{Var}[\mathbf{R}_i] \leq \frac{n}{k}.
\end{aligned}$$

n: total number of requests, k: number of servers randomly assigned requests, $\mathbf{R}_i$: number of requests assigned to server i.

Letting R_i be the number of requests sent to server i, $\mathbb{E}[\mathsf{R}_i] = \frac{n}{k}$ and $\text{Var}[\mathsf{R}_i] \le \frac{n}{k}$.

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Applying Chebyshev's:

$$\Pr\left(\mathsf{R}_i \geq \frac{2n}{k}\right) \leq \Pr\left(|\mathsf{R}_i - \mathbb{E}[\mathsf{R}_i]| \geq \frac{n}{k}\right) \leq \frac{\text{Var}(R_i)}{n^2/k^2}$$

$$\Pr\left((R_i - \mathbb{E}R_i) \geq \frac{n}{k} \text{ or } (R_i - \mathbb{E}R_i) \leq -\frac{n}{k}\right)$$

$$\leq \frac{n/k}{n^2/k^2} = \frac{k}{n}$$

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BOUNDING THE LOAD VIA CHEBYSHEVS

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Chebyshevs inequality

Applying Chebyshev's:

$$\Pr\left(\mathsf{R}_i \geq \frac{2n}{k}\right) \leq \Pr\left(|\mathsf{R}_i - \mathbb{E}[\mathsf{R}_i]| \geq \frac{n}{k}\right) \leq \frac{n/k}{n^2/k^2} = \frac{k}{n}.$$

$\frac{1}{2}$

- Overload probability is extremely small when $k \ll n$!

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- Overload probability is extremely small when $k \ll n$!
- Might seem counterintuitive – bound gets worse as k grows.
- When k is large, the number of requests each server sees in expectation is very small so the law of large numbers doesn't 'kick in'.

n: total number of requests, k: number of servers randomly assigned requests, R_i: number of requests assigned to server i.

What is the probability that the maximum server load exceeds $2 \cdot \mathbb{E}[R_i] = \frac{2n}{k}$. I.e., that some server is overloaded if we give each $\frac{2n}{k}$ capacity?

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MAXIMUM SERVER LOAD

What is the probability that the maximum server load exceeds $2 \cdot \mathbb{E}[\mathrm{R}_i] = \frac{2n}{k}$. I.e., that some server is overloaded if we give each $\frac{2n}{k}$ capacity?

$$\Pr\left(\max_i(\mathrm{R}_i) \geq \frac{2n}{k}\right)$$

n: total number of requests, k: number of servers randomly assigned requests, R_i: number of requests assigned to server i. $\mathbb{E}[\mathrm{R}_i] = \frac{n}{k}$. $\mathrm{Var}[\mathrm{R}_i] = \frac{n}{k}$.

What is the probability that the maximum server load exceeds $2 \cdot \mathbb{E}[R_i] = \frac{2n}{k}$. I.e., that some server is overloaded if we give each $\frac{2n}{k}$ capacity?

$$\Pr\left(\max_i(R_i) \geq \frac{2n}{k}\right) = \Pr\left(\left[R_1 \geq \frac{2n}{k}\right] \cup \left[R_2 \geq \frac{2n}{k}\right] \cup \ldots \cup \left[R_k \geq \frac{2n}{k}\right]\right)$$

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$$\Pr\left(\max_i(R_i) \geq \frac{2n}{k}\right) = \Pr\left(\left[R_1 \geq \frac{2n}{k}\right] \text{ or } \left[R_2 \geq \frac{2n}{k}\right] \text{ or } \ldots \text{ or } \left[R_k \geq \frac{2n}{k}\right]\right)$$

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$$\Pr\left(\max_i(\mathbf{R}_i) \geq \frac{2n}{k}\right) = \Pr\left(\bigcup_{i=1}^{k}\left[\mathbf{R}_i \geq \frac{2n}{k}\right]\right)$$

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How do we do this? Note that $\mathrm{R}_1, \ldots, \mathrm{R}_k$ are correlated in a somewhat complex way.

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Union Bound: For any random events $A_1, A_2, ..., A_k$,

$$\Pr(A_1 \cup A_2 \cup \ldots \cup A_k) \leq \Pr(A_1) + \Pr(A_2) + \ldots + Pr(A_k).$$

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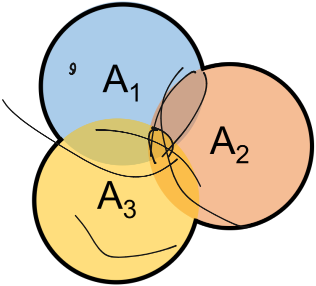

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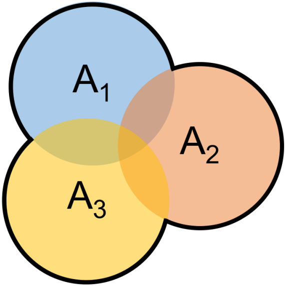

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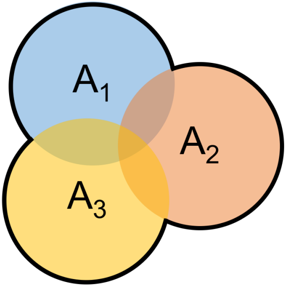

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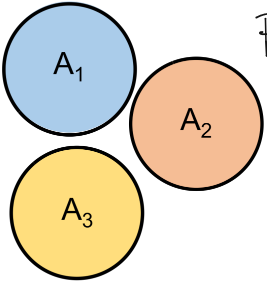

$$P(D=1 \text{ or } D=2 \text{ or } D=3) = \frac{1}{2} = \frac{1}{6} + \frac{1}{6} + \frac{1}{6}$$

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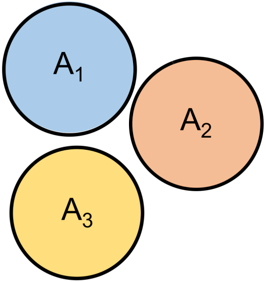

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On the first problem set, you will prove the union bound, as a consequence of Markov's inquality.

APPLYING THE UNION BOUND

What is the probability that the maximum server load exceeds $2 \cdot \mathbb{E}[\mathbf{R}_i] = \frac{2n}{k}$. I.e., that some server is overloaded if we give each $\frac{2n}{k}$ capacity?

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As long as $k \leq O(\sqrt{n})$, with good probability, the maximum server load will be small (compared to the expected load).

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- There are many requests routed to a relatively small number of servers so the load seen on each server is close to what is expected via law of large numbers.
- **A More Natural Variant:** Given n requests, and assuming all servers have fixed capacity C, how many servers should you provision so that with probability $\geq 99/100$ no server is assigned more than C requests?

n: total number of requests, k: number of servers randomly assigned requests.

Questions on union bound, Chebyshev's inequality, random hashing?

We flip $n = 100$ independent coins, each are heads with probability 1/2 and tails with probability 1/2. Let $\mathbf{H}$ be the number of heads.

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Markov's:

$\Pr(\mathsf{H} \geq 60) \leq .833$

$\Pr(\mathsf{H} \geq 70) \leq .714$

$\Pr(\mathsf{H} \geq 80) \leq .625$

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Markov's:	Chebyshev's:
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Markov's:	Chebyshev's:	In Reality:
$\Pr(H \geq 60) \leq .833$	$\Pr(H \geq 60) \leq .25$	$\Pr(H \geq 60) = 0.0284$
$\Pr(H \geq 70) \leq .714$	$\Pr(H \geq 70) \leq .0625$	$\Pr(H \geq 70) = .000039$
$\Pr(H \geq 80) \leq .625$	$\Pr(H \geq 80) \leq .0278$	$\Pr(H \geq 80) < 10^{-9}$

H has a simple Binomial distribution, so can compute these probabilities exactly.

To be fair.... Markov and Chebyshev's inequalities apply much more generally than to Binomial random variables like coin flips.

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Can we obtain tighter concentration bounds that still apply to very general distributions?