COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

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University of Massachusetts Amherst. Fall 2021.

Lecture 2

Reminders:

- Sign up for Piazza – there has already been a lot of great discussion.
- Find homework teammates and sign up for Gradescope (code on course website).
- My office hours (on Zoom) have moved to Thursday, 9:00am-10:30am.

Last Class We Covered:

- Basic probability review. See course site for links to resources to refresh your probability background.
- Linearity of expectation: $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$ always.
- Linearity of variance: $\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y]$ if X and Y are independent.

Today:

- An algorithmic application of of linearity of expectation and variance.
- Introduce Markov's inequality a fundamental concentration bound that let us prove that a random variable lies close to its expectation with good probability.
- Learn about random hash functions, which are a key tool in randomized methods for data processing. Probabilistic analysis via linearity of expectation.

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What is $Var(X/3)$?

The expected number of inches of rain on Saturday is 6 and the expected number of inches on Sunday is 5. There is a 50% chance of rain on Saturday. If it rains on Saturday, there is a 75% chance of rain on Sunday. If it does not rain on Saturday, there is only a 25% chance of rain on Sunday. What is the expected number of inches of rainfall total over the weekend?

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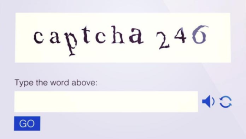

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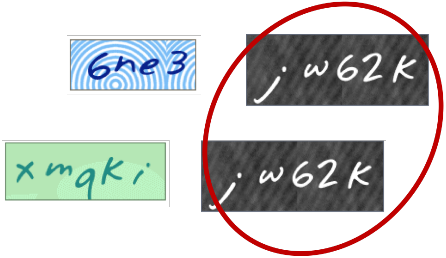
- You want to independently verify this claimed database size.
- You could make test checks until you see $1,000,000$ unique CAPTCHAS: would take $\geq 1,000,000$ checks!

An Idea: You run some test security checks and see if any duplicate CAPTCHAS show up. If you're seeing duplicates after not too many checks, the database size is probably not too big.



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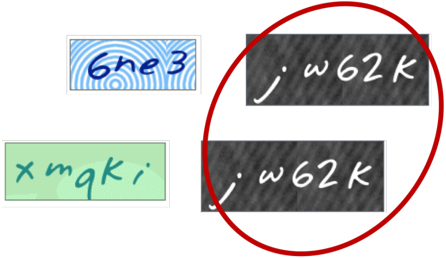

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AN ALGORITHMIC APPLICATION

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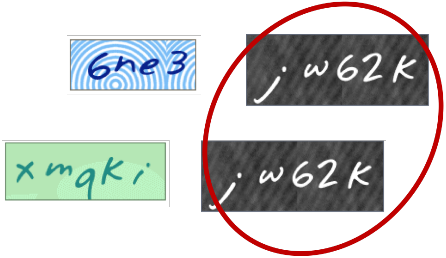

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Think-Pair-Share: If you run m security checks, and there are n unique CAPTCHAS, how many pairwise duplicates do you see in expectation?

If e.g. the same CAPTCHA shows up three times, on your i^{th}, j^{th}, and k^{th} test, this is three duplicates: (i, j), (i, k) and (j, k).

Let $\mathbf{D}_{i,j} = 1$ if tests i and j give the same CAPTCHA, and 0 otherwise. An indicator random variable.

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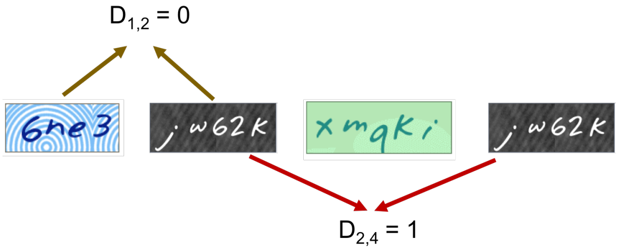

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Note that the $\mathbf{D}_{i,j}$ random variables are not independent!

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$$\mathbb{E}[D] = \frac{m(m-1)}{2n} = \frac{110 \cdot 109}{2 \cdot 365} \approx 16.5.$$

You take $m = 1000$ samples. If the database size is as claimed ($n = 1,000,000$) then expected number of duplicates is:

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Concentration Inequalities: Bounds on the probability that a random variable deviates a certain distance from its mean.

- Useful in understanding how statistical tests perform, the behavior of randomized algorithms, the behavior of data drawn from different distributions, etc.

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The larger the deviation t, the smaller the probability.

Expected number of duplicate CAPTCHAS:

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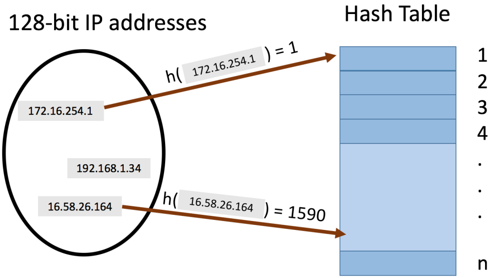
- *Static hashing* since we won't worry about insertion and deletion today.



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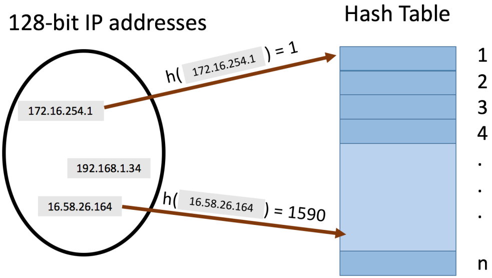

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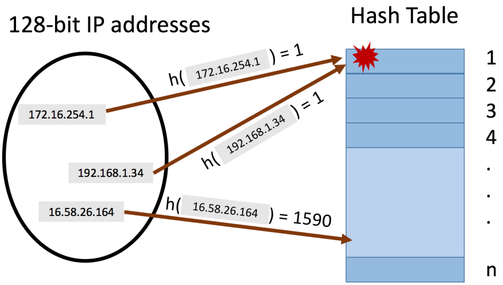

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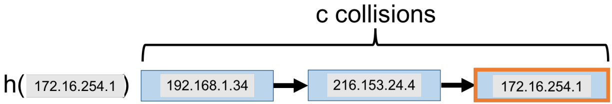
- Typically $|U| \gg n$. Many elements map to the same index.
- **Collisions:** when we insert m items into the hash table we may have to store multiple items in the same location (typically as a linked list).

Query runtime: $O(c)$ when the maximum number of collisions in a table entry is c (i.e., must traverse a linked list of size c).



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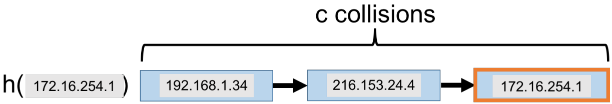

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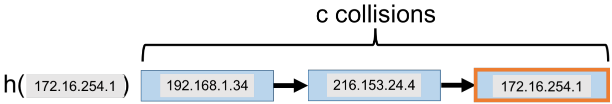

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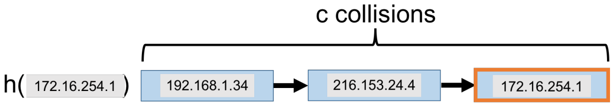

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- Two approaches: 1) we assume the items inserted are chosen randomly from the universe U or 2) we assume the hash function is random.

Let $\mathbf{h} : U \to [n]$ be a fully random hash function.

- I.e., for $x \in U$, $\Pr(\mathbf{h}(x) = i) = \frac{1}{n}$ for all $i = 1, \ldots, n$ and $\mathbf{h}(x), \mathbf{h}(y)$ are independent for any two items $x \neq y$.

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- **Caveat 1:** It is *very expensive* to represent and compute such a random function. We will see how a hash function computable in $O(1)$ time function can be used instead.
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Think-Pair-Share: Assuming we insert m elements into a hash table of size n, what is the expected total number of pairwise collisions?

Let $\mathbf{C}_{i,j} = 1$ if items i and j collide $(\mathbf{h}(x_i) = \mathbf{h}(x_j))$, and 0 otherwise. The number of pairwise duplicates is:

$$\mathbf{C} = \sum_{i,j \in [m], i \neq j} \mathbf{C}_{i,j}.$$

x_i, x_j: pair of stored items, m: total number of stored items, n: hash table size, $\mathbf{C}$: total pairwise collisions in table, $\mathbf{h}$: random hash function.

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$$\mathbb{E}[\mathsf{C}] = \sum_{i,j \in [m], i \neq j} \frac{1}{n} = \frac{\binom{m}{2}}{n} = \frac{m(m-1)}{2n}.$$

Identical to the CAPTCHA analysis!

x_i, x_j: pair of stored items, m: total number of stored items, n: hash table size, C: total pairwise collisions in table, $\mathbf{h}$: random hash function.

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Apply Markov's Inequality: $\Pr[\mathbf{C} \geq 1] \leq \frac{\mathbb{E}[\mathbf{C}]}{1}$

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- **Think-Pair-Share:** Give a lower bound on the probability that we have no collisions, i.e., $\Pr[\mathbf{C} = 0]$?

Apply Markov's Inequality: $\Pr[\mathbf{C} \geq 1] \leq \frac{\mathbb{E}[\mathbf{C}]}{1} = \frac{1}{8}$.

$$\Pr[\mathbf{C} = 0] = 1 - \Pr[\mathbf{C} \geq 1] \geq 1 - \frac{1}{8} = \frac{7}{8}.$$

Pretty good...but we are using $O(m^2)$ space to store m items...

> m: total number of stored items, n: hash table size, $\mathbf{C}$: total pairwise collisions in table.

Questions?