COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

Prof. Cameron Musco

University of Massachusetts Amherst. Fall 2021.

Lecture 1

FACE MASKS

Masks covering your nose and mouth are required in class (and generally indoors at UMass) regardless of vaccination status.

- There is no exception for eating/drinking, so if you need to take a drink, please step outside briefly to do so.

MOTIVATION FOR THIS CLASS

People are increasingly interested in analyzing and learning from massive datasets.

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  - How do they process them to target advertisements? To predict trends? To improve their products?

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- The Large Synoptic Survey Telescope will take high definition photographs of the sky, producing 15 terabytes of data/night.
  - How do they denoise and compress the images? How do they detect anomalies such as changing brightness or position of objects to alert researchers?

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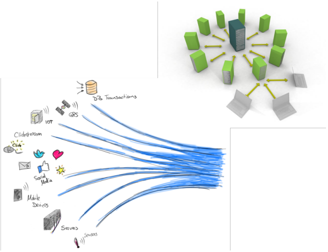

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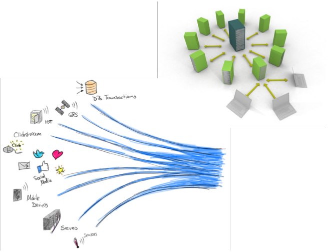

- Even ‘simple’ problems become very difficult in this setting.

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- How can Google estimate the number of unique search queries that are made in a given week? Given that no machine can store the full list of queries.
- When you use Shazam to identify a song from a recording, how does it provide an answer in < 10 seconds, without scanning over all $\sim$ 8 million audio files in its database.

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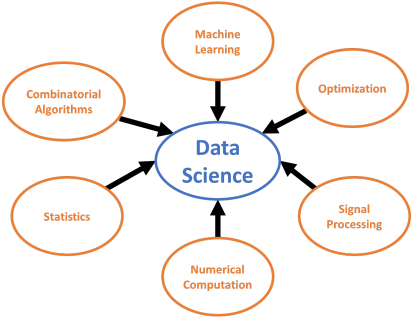

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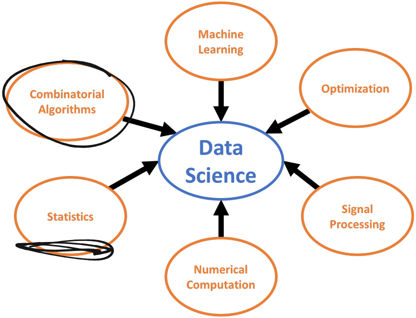

- Many techniques that aren't covered in the traditional CS algorithms curriculum.
- Emphasis on building comfort with mathematical tools that underly data science and machine learning.

WHAT WE'LL COVER

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- Probability tools and concentration inequalities.
- Randomized hashing for efficient lookup, load balancing, and estimation. Bloom filters.
- Locality sensitive hashing and nearest neighbor search.
- Streaming algorithms: identifying frequent items in a data stream, counting distinct items, etc.
- Random compression of high-dimensional vectors: the Johnson-Lindenstrauss lemma, applications, and connections to the weirdness of high-dimensional geometry.

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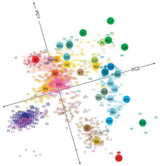

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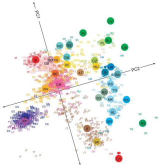

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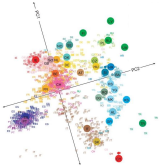

How do we identify the most important features of a dataset using linear algebraic techniques?

- Principal component analysis, low-rank approximation, dimensionality reduction.
- The singular value decomposition (SVD) and its applications to PCA, low-rank approximation, LSI, MDS, …
- Spectral graph theory. Spectral clustering, community detection, network visualization.
- Computing the SVD on large datasets via iterative methods.

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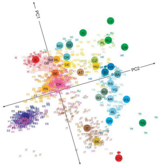

How do we identify the most important features of a dataset using linear algebraic techniques?

If you open up the codes that are underneath [most data science applications] this is all linear algebra on arrays.

– Michael Stonebraker

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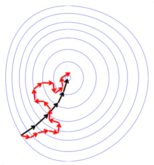
Section 3: Optimization



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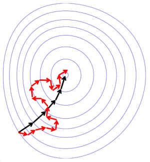

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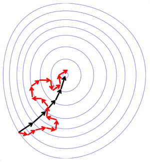

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- Stochastic and online gradient descent.
- Focus on convergence analysis.

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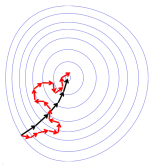

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A small taste of what you can find in COMPSCI 590OP or 690OP.

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  - E.g., regression methods, kernel methods, random forests, SVM, deep neural networks.

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  - COMPSCI 589/689: Machine Learning

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For example: Baye's rule in conditional probability. What it means for a vector x to be an eigenvector of a matrix A, orthogonal projection, greedy algorithms, divide-and-conquer algorithms.

See course webpage for logistics, policies, lecture notes, assignments, etc.:

http://people.cs.umass.edu/~cmusco/CS514F21/

See Moodle page for this link if you lose it.

Professor: Cameron Musco

- Email: cmusco@cs.umass.edu
- Office Hours: Over Zoom, Tuesdays, 2:30pm-3:30pm (directly after class). See website for Zoom link. ?
- I encourage you to come as regularly as possible to ask questions/work together on practice problems.
- If you need to chat individually, please email meet to set up a time.

TAs:

- Pratheba Selvaraju
- Shiv Shankar
- Weronika Nguyen

See website for office hours and contact info.

There is also an online version of 514 taught this semester by Andrew McGregor.

- The sections will closely parallel each other, and share the same TAs.
- You may attend Prof. McGregor's lectures and office hours (both over Zoom) if it is helpful.
- See course webpage for schedule and Moodle for Zoom links.

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You may earn up to 5% extra credit for participation.

- Asking good clarifying questions and answering questions during the lecture or on Piazza.
- Actively participating in office hours.
- Answering other students' or instructor questions on Piazza.
- Posting helpful/interesting links on Piazza, e.g., resources that cover class material, research articles related to the topics covered in class, etc.

We will use material from two textbooks (links to free online versions on the course webpage): *Foundations of Data Science* and *Mining of Massive Datasets*, but will follow neither closely.

- I will post optional readings a few days prior to each class.
- Lecture notes will be posted before each class, and annotated notes posted after class.
- Recordings of the live lectures will also be posted.

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HOMEWORK

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- We strongly encourage working in groups, as it will make completing the problem sets much easier/more educational.
- Collaboration with students outside your group is limited to discussion at a high level. You may not work through problems in detail or write up solutions together.
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Problem set submissions will be via Gradescope.

- See website for a link to join. **Entry Code: P5NKXN**

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- Designed as a check-in that you are following the material, and to help me make adjustments as needed.
- Will take around 15-30 minutes per week, open notes.
- Will also include free response check-in questions to get your feedback on how the course is going, what material from the past week you find most confusing, interesting, etc.

Grade Breakdown:

- Problem Sets (5 total): 40%, weighted equally.
- Weekly Quizzes: 10%, weighted equally.
- Midterm (October 19th, in class): 25%.
- Final (December 16th, 10:30am - 12:30pm): 25%.
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Academic Honesty:

- A first violation cheating on a homework, quiz, or other assignment will result in a 0 on that assignment.
- A second violation, or cheating on an exam will result in failing the class.
- For fairness, I adhere very strictly to these policies.

DISABILITY SERVICES AND ACCOMODATIONS

UMass Amherst is committed to making reasonable, effective, and appropriate accommodations to meet the needs to students with disabilities.

- If you have a documented disability **on file with Disability Services**, you may be eligible for reasonable accommodations in this course.
- If your disability requires an accommodation, please email me by **next Thursday 9/9** so that we can make arrangements.

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I understand that people have different learning needs, home situations, etc. If something isn't working for you in the class, please reach out and let's try to work it out.

Questions?

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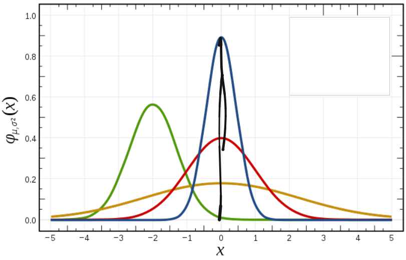
- **Expectation:** $\mathbb{E}[\mathbf{X}] = \sum_{s \in S} \Pr(\mathbf{X} = s) \cdot s$.

$E[X] = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \ldots \frac{1}{6} \cdot 6 = 3.5$



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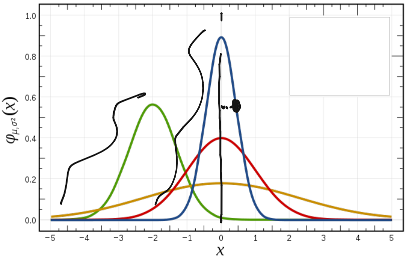

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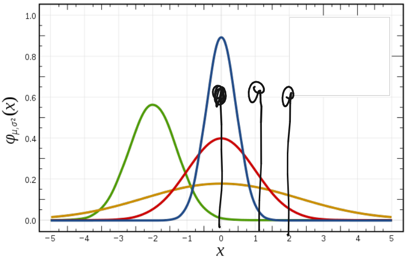

Exercise: Show that for any scalar α, $\mathbb{E}[\alpha \cdot \mathbf{X}] = \alpha \cdot \mathbb{E}[\mathbf{X}]$ and $\text{Var}[\alpha \cdot \mathbf{X}] = \alpha^2 \cdot \text{Var}[\mathbf{X}]$.

Consider two random events A and B.

$A \cap B$: event that both events *A and B* happen.

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Using the definition of conditional probability, independence means:

$$\frac{\Pr(A \cap B)}{\Pr(B)} = \Pr(A) \implies \Pr(A \cap B) = \Pr(A) \cdot \Pr(B).$$

$A \cap B$: event that both events A *and* B happen.

For Example: What is the probability that for two independent dice rolls the first is a 6 and the second is odd?

$$P(D_1 = 6 \cap D_2 \in \{1,3,5\})$$

$$P(D_1 = 6) \cdot P(D_2 \in \{1,3,5\})$$

$$\frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}$$

Independent Random Variables: Two random variables X, Y are independent if for all s, t, $X = s$ and $Y = t$ are independent events. In other words:

independent: $\Pr(X = s \cap Y = t) = \Pr(X = s) \cdot \Pr(Y = t)$.

uncorrelated: $\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$

Think-Pair-Share: When are the expectation and variance linear?

I.e., under what conditions on X and Y do we have:

$$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

and

$$\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y].$$

X, Y: any two random variables.

- ~~when independent~~

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$$= \sum_{s \in S} \sum_{t \in T} \Pr(X = s \cap Y = t) \cdot s + \sum_{t \in T} \sum_{s \in S} \Pr(X = s \cap Y = t) \cdot t$$

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$$= \sum_{s \in S} \Pr(X = s) \cdot s + \sum_{t \in T} \Pr(Y = t) \cdot t$$

$\mathbb{E}[X] + \mathbb{E}[Y]$ (law of total probability)

$\mathbb{E}[X+Y] = \frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 2 = 1$

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Proof:

$$\begin{aligned}
\mathbb{E}[X+Y] &= \sum_{s\in S}\sum_{t\in T} \Pr(X = s \cap Y = t)\cdot(s+t) \\
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&= \sum_{s\in S} \Pr(X = s)\cdot s + \sum_{t\in T} \Pr(Y = t)\cdot t
\end{aligned}$$

(law of total probability)

$$= \mathbb{E}[X] + \mathbb{E}[Y].$$

$X = 0$ w.p $\frac{1}{2}$, 1 w.p $\frac{1}{2}$

$Y = 0$ w.p $\frac{1}{2}$, 1 w.p. $\frac{1}{2}$

$\mathbb{E}[X+Y] = \frac{1}{4}\cdot 0 + \frac{1}{4}\cdot 1 + \frac{1}{4}\cdot 1 + \frac{1}{4}\cdot 2$

$= 1$

$$\mathrm{Var}[X + Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$$ → when X and Y are independent

$Y = X$

$\mathrm{Var}(X+Y) = \mathrm{Var}(2X) = 4 \cdot \mathrm{Var}(X)$

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Claim 1: (exercise) $\text{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$ (via linearity of expectation)

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Together give: $\mathbb{E}[(X^2 + 2XY + Y^2)]$

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Questions?

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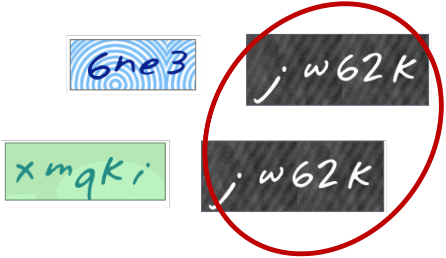
- You want to independently verify this claimed database size.
- You could make test checks until you see $1,000,000$ unique CAPTCHAS: would take $\geq 1,000,000$ checks!

AN ALGORITHMIC APPLICATION

An Idea: You run some test security checks and see if any duplicate CAPTCHAS show up. If you're seeing duplicates after not too many checks, the database size is probably not too big.



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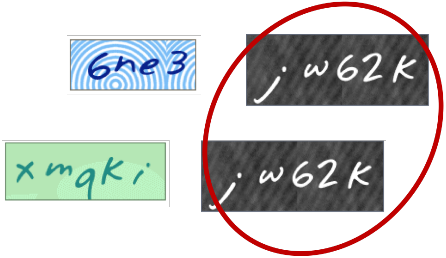

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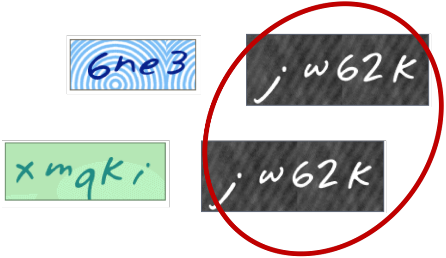

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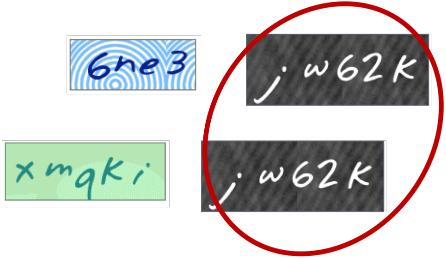

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If e.g. the same CAPTCHA shows up three times, on your i^{th}, j^{th}, and k^{th} test, this is three duplicates: (i, j), (i, k) and (j, k).

Let $\mathbf{D}_{i,j} = 1$ if tests i and j give the same CAPTCHA, and 0 otherwise. An indicator random variable.

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Note that the $\mathbf{D}_{i,j}$ random variables are not independent!

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You take $m = 1000$ samples. If the database size is as claimed ($n = 1,000,000$) then expected number of duplicates is:

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Concentration Inequalities: Bounds on the probability that a random variable deviates a certain distance from its mean.

- Useful in understanding how statistical tests perform, the behavior of randomized algorithms, the behavior of data drawn from different distributions, etc.

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