COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

Cameron Musco

University of Massachusetts Amherst. Fall 2020.

Lecture 9

- Problem Set 2 is due this upcoming Monday. Get an early start on it.
- Problem Set 1 grades have been released. Mean: 34/41, Median 36/41.
- If you are unhappy with your grade, ping me and let's chat about strategies going forward. If you believe there is a grading error, send a private message to the instructors on Piazza or ask during office hours.
- The midterm will be any 2 hour slot on 10/8-10/9. We won't have class on 10/8.
- Study guide/practice questions will be released this week.

SUMMARY

Last Class:

- MinHash as a locality sensitive hash function for Jaccard similarity
- Near neighbor search with LSH signatures and repeated hash tables..
- SimHash for cosine similarity.

A locality sensitive hash function can be: (check all that apply)

Select one or more:

- [ ] a. Randomized
- [ ] b. Pairwise-Independent
- [ ] c. Sensitive to Jaccard Similarity
- [ ] d. Have the distribution of h(x) independent of x.

Next Few Classes:

- Random compression methods for high dimensional vectors. The Johnson-Lindenstrauss lemma.
- Connections to the weird geometry of high-dimensional space.

After That: Spectral Methods

- PCA, low-rank approximation, and the singular value decomposition.
- Spectral clustering and spectral graph theory.

Will use a lot of linear algebra. May be helpful to refresh.

- Vector dot product, addition, Euclidean norm. Matrix vector multiplication.
- Linear independence, column span, orthogonal bases, rank.
- Orthogonal projection, eigendecomposition, linear systems.

k-**Frequent Items (Heavy-Hitters) Problem**: Consider a stream of n items $x_1, \ldots, x_n$ (with possible duplicates). Return any item at appears at least $\frac{n}{k}$ times.

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_1
5	12	3	3	4	5	5	10	3	5

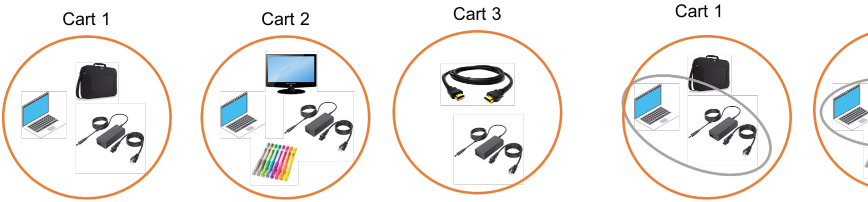
- What is the maximum number of items that must be returned? a) n b) k c) n/k d) $\log n$
- Trivial with $O(n)$ space – store the count for each item and return the one that appears $\geq n/k$ times.
- Can we do it with less space? I.e., without storing all n items?

Applications of Frequent Items:

- Finding top/viral items (i.e., products on Amazon, videos watched on Youtube, Google searches, etc.)
- Finding very frequent IP addresses sending requests (to detect DoS attacks/network anomalies).
- 'Iceberg queries' for all items in a database with frequency above some threshold.

Generally want very fast detection, without having to scan through database/logs. I.e., want to maintain a running list of frequent items that appear in a stream.

Association rule learning: A very common task in data mining is to identify common associations between different events.



- Identified via frequent itemset counting. Find all sets of k items that appear many times in the same basket.
- Frequency of an itemset is known as its support.
- A single basket includes many different itemsets, and with many different baskets an efficient approach is critical. E.g., baskets are Twitter users and itemsets are subsets of who they follow.

Issue: No algorithm using $o(n)$ space can output just the items with frequency $\geq n/k$. Hard to tell between an item with frequency n/k (should be output) and $n/k - 1$ (should not be output).

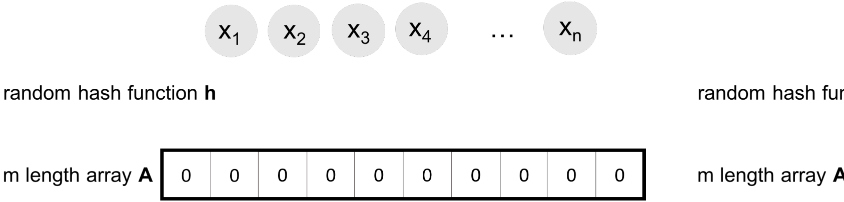
x_1	x_2	x_3	x_4	x_5	x_6	...	$x_{n-n/k+1}$	...	x_n
3	12	9	27	4	101		3		3

n/k-1 occurrences

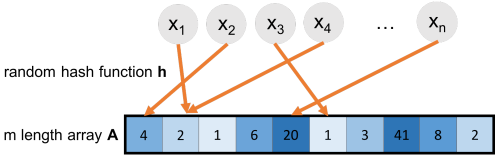
(ϵ, k)-Frequent Items Problem: Consider a stream of n items $x_1, \ldots, x_n$. Return a set F of items, including all items that appear at least $\frac{n}{k}$ times and only items that appear at least $(1-\epsilon) \cdot \frac{n}{k}$ times.

- An example of relaxing to a 'promise problem': for items with frequencies in $[(1-\epsilon) \cdot \frac{n}{k}, \frac{n}{k}]$ no output guarantee.

Today: Count-min sketch – a random hashing based method closely related to bloom filters.



Will use $A[\mathbf{h}(x)]$ to estimate $f(x)$, the frequency of x in the stream. I.e., $|\{x_i : x_i = x\}|$.



Use $A[\mathbf{h}(x)]$ to estimate $f(x)$.

Claim 1: We always have $A[\mathbf{h}(x)] \geq f(x)$. Why?

- $A[\mathbf{h}(x)]$ counts the number of occurrences of any y with $\mathbf{h}(y) = \mathbf{h}(x)$, including x itself.
- $A[\mathbf{h}(x)] = f(x) + \sum_{y \neq x : \mathbf{h}(y) = \mathbf{h}(x)} f(y)$.

$f(x)$: frequency of x in the stream (i.e., number of items equal to x). $\mathbf{h}$: random hash function. m: size of Count-min sketch array.

$$A[\mathbf{h}(x)] = f(x) + \underbrace{\sum_{y \neq x : \mathbf{h}(y) = \mathbf{h}(x)} f(y)}_{\text{error in frequency estimate}}.$$

Expected Error:

$$\mathbb{E}\left[\sum_{y \neq x : \mathbf{h}(y) = \mathbf{h}(x)} f(y)\right] = \sum_{y \neq x} \Pr(\mathbf{h}(y) = \mathbf{h}(x)) \cdot f(y)$$

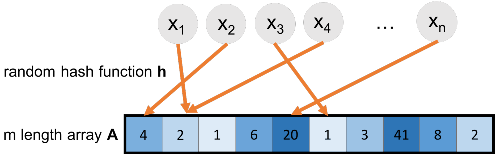
$$= \sum_{y \neq x} \frac{1}{m} \cdot f(y) = \frac{1}{m} \cdot (n - f(x)) \leq \frac{n}{m}$$

What is a bound on probability that the error is $\geq \frac{2n}{m}$?

Markov's inequality: $\Pr\left[\sum_{y \neq x : \mathbf{h}(y) = \mathbf{h}(x)} f(y) \geq \frac{2n}{m}\right] \leq \frac{1}{2}$.

What property of **h** is required to show this bound? a) fully random b) pairwise independent c) 2-universal d) locality sensitive

> $f(x)$: frequency of x in the stream (i.e., number of items equal to x). $\mathbf{h}$: random hash function. m: size of Count-min sketch array.



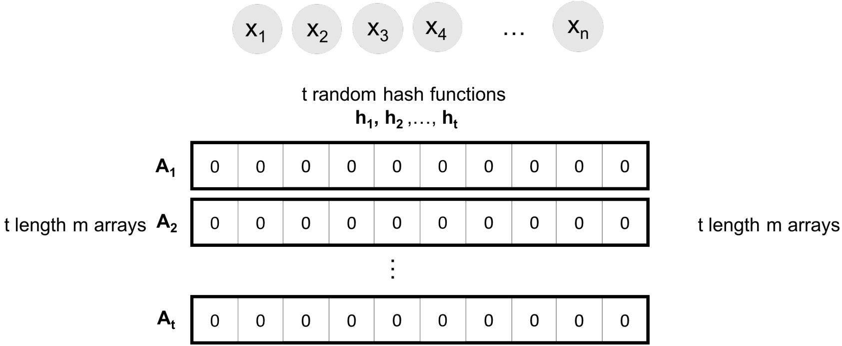
Claim: For any x, with probability at least 1/2,

$$f(x) \leq A[\mathbf{h}(x)] \leq f(x) + \frac{2n}{m}.$$

To solve the (ϵ, k)-Frequent elements problem, set $m = \frac{2k}{\epsilon}$.

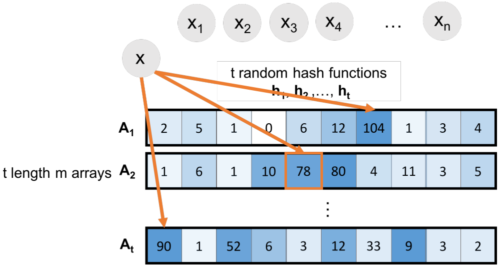
How can we improve the success probability? Repetition.

> $f(x)$: frequency of x in the stream (i.e., number of items equal to x). $\mathbf{h}$: random hash function. m: size of Count-min sketch array.



Estimate $f(x)$ with $\tilde{f}(x) = \min_{i \in [t]} A_i[\mathbf{h}_i(x)]$. (count-min sketch)

Why min instead of mean or median? The minimum estimate is always the most accurate since they are all overestimates of the true frequency!



Estimate $f(x)$ by $\tilde{f}(x) = \min_{i \in [t]} A_i[\mathbf{h}_i(x)]$

- For every x and $i \in [t]$, we know that for $m = \frac{2k}{\epsilon}$, with probability $\geq 1/2$:

$$f(x) \leq A_i[\mathbf{h}_i(x)] \leq f(x) + \frac{\epsilon n}{k}.$$

- What is $\Pr[f(x) \leq \tilde{f}(x) \leq f(x) + \frac{\epsilon n}{k}]$? $1 - 1/2^t$.
- To get a good estimate with probability $\geq 1 - \delta$, set $t = \log(1/\delta)$.

Upshot: Count-min sketch lets us estimate the frequency of every item in a stream up to error $\frac{\epsilon n}{k}$ with probability $\geq 1 - \delta$ in $O\left(\log(1/\delta) \cdot k/\epsilon\right)$ space.

- Accurate enough to solve the (ϵ, k)-Frequent elements problem – distinquish between items with frequency $\frac{n}{k}$ and those with frequency $(1 - \epsilon)\frac{n}{k}$.
- How should we set δ if we want a good estimate for all items at once, with 99% probability?

Count-min sketch gives an accurate frequency estimate for every item in the stream. But how do we identify the frequent items without having to store/look up the estimated frequency for all elements in the stream?

One approach:

- When a new item comes in at step i, check if its estimated frequency is $\geq i/k$ and store it if so.
- At step i remove any stored items whose estimated frequency drops below i/k.
- Store at most $O(k)$ items at once and have all items with frequency $\geq n/k$ stored at the end of the stream.

Questions on Frequent Elements?