COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

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University of Massachusetts Amherst. Fall 2020.

Lecture 6

LOGISTICS

- Problem Set 1 is due tomorrow at 8pm in Gradescope.
- Quiz 3 will be due next Monday at 8pm on Moodle.

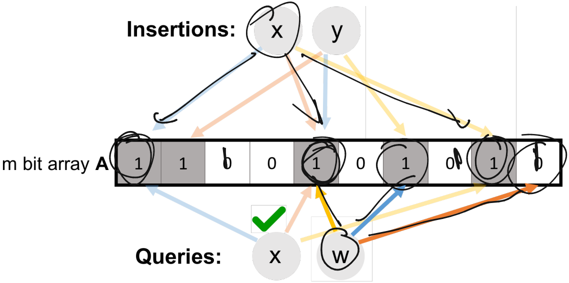
Last Class:

- Exponential concentration bound wrap up (central limit theorem, Chernoff bound).
- Bloom Filters:
  - Random hashing to maintain a large set in small space.
  - Discussed applications and how the false positive rate is determined.

This Class:

- Wrap up Bloom filters.
- Start on streaming algorithms – distinct items counting.

m-bit array. Each inserted item is marked with k bits, determined by k random hash functions.



- $query(x) = 1$ if and only if all bits that x hashes to are 1 (i.e., $A[h_1(x)] = \ldots = A[h_k(x)] = 1$.)
- Can be false positives, but no false negatives.

How does the false positive rate δ depend on m, k, and the number of items inserted n?

Step 1: What is the probability that after inserting n elements, the i^{th} bit of the array A is still 0?

$$\Pr(A[i] = 0) = \left(1 - \frac{1}{m}\right)^{kn} \approx e^{-\frac{kn}{m}}$$

n: total number items in filter, m: number of bits in filter, k: number of random hash functions, $\mathbf{h}_1, \ldots \mathbf{h}_k$: hash functions, A: bit array, δ: false positive rate.

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$$\Pr\left(A[\mathbf{h}_1(w)] = \ldots = A[\mathbf{h}_k(w)] = 1\right)$$
$$= \Pr(A[\mathbf{h}_1(w)] = 1) \times \ldots \times \Pr(A[\mathbf{h}_k(w)] = 1)$$

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$$\begin{aligned}\Pr\left(A[\mathbf{h}_1(w)] = \ldots = A[\mathbf{h}_k(w)] = 1\right) &= \Pr(A[\mathbf{h}_1(w)] = 1) \times \ldots \times \Pr(A[\mathbf{h}_k(w)] = 1) \\ &= \left(1 - e^{-\frac{kn}{m}}\right)^k\end{aligned}$$

Actually Incorrect! Dependent events.

> n: total number items in filter, m: number of bits in filter, k: number of random hash functions, $\mathbf{h}_1, \ldots \mathbf{h}_k$: hash functions, A: bit array, δ: false positive rate.

CORRECT ANALYSIS SKETCH

Step 1: To avoid dependence issues, condition on the event that the A has t zeros in it after n insertions, for some $t \leq m$. For a non-inserted element w, after conditioning on this event we correctly have:

$$\Pr(A[\mathbf{h}_1(w)] = \ldots = A[\mathbf{h}_k(w)] = 1)$$
$$= \Pr(A[\mathbf{h}_1(w)] = 1) \times \ldots \times \Pr(A[\mathbf{h}_k(w)] = 1).$$

I.e., the events $A[\mathbf{h}_1(w)] = 1, \ldots, A[\mathbf{h}_k(w)] = 1$ are independent conditioned on the number of bits set in A. Why?

- Conditioned on this event, for any j, since $\mathbf{h}_j$ is a fully random hash function, $\Pr(A[\mathbf{h}_j(w)] = 1) = \frac{t}{m}$.
- Thus conditioned on this event, the false positive rate is $\left(1 - \frac{t}{m}\right)^k$.
- It remains to show that $\frac{t}{m} \approx e^{-\frac{kn}{m}}$ with high probability. We already have that $\mathbb{E}[\frac{t}{m}] = \frac{1}{m}\sum_{i=1}^{m} \Pr(A[i] = 0) \approx e^{-\frac{kn}{m}}$.

Need to show that the number of zeros t in A after n insertions is bounded by $O\left(e^{-\frac{kn}{m}}\right)$ with high probability.

Can apply Theorem 2 of: `http://cglab.ca/~morin/publications/ds/bloom-submitted.pdf`

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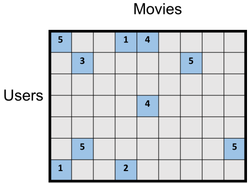

- We have 100 million users and $10,000$ movies. On average each user has rated only 10 movies so of these 10^{12} possible (user,movie) pairs, only $10 * 100,000,000 = 10^9 = n$ (user,movie) pairs have non-empty entries in our table.
- We allocate $m = 8n = 8 \times 10^9$ bits for a Bloom filter (1 GB).

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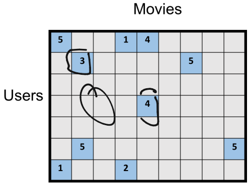

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- We allocate $m = 8n = 8 \times 10^9$ bits for a Bloom filter (1 GB). How should we set k to minimize the number of false positives?

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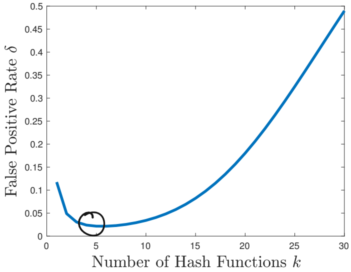
$n = 10^9$

$m = 8 \cdot 10^9$



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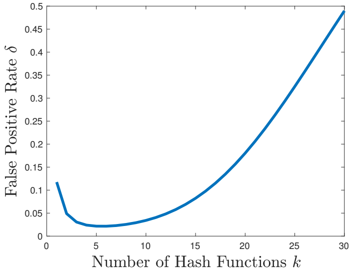

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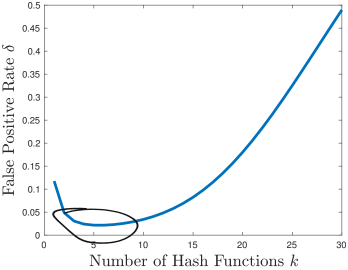
(fraction of 1 entries)^k

$1 - e^{-\frac{\ln 2 \cdot \frac{m}{n} \cdot n}{m}}$

$1 - e^{-\ln 2} = \frac{1}{2}$



- Can differentiate to show optimal number of hashes is $k = \ln 2 \cdot \frac{m}{n}$.
- Balances between filling up the array with too many hashes and having enough hashes so that even when the array is pretty full, a new item is unlikely to have all its bits set (yield a false positive)

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I.e., storing n items in a bloom filter requires $O(n)$ space. So what's the point? Truly $O(n)$ bits, rather than $O(n \cdot \text{item size})$.

Questions on Bloom Filters?

Stream Processing: Have a massive dataset X with n items $x_1, x_2, \ldots, x_n$ that arrive in a continuous stream. Not nearly enough space to store all the items (in a single location).

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- Typically must compress the data on the fly, storing a data structure from which you can still learn useful information.
- Often the compression is randomized. E.g., bloom filters.
- Compared to traditional algorithm design, which focuses on minimizing runtime, the big question here is how much space is needed to answer queries of interest.

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Breakout Rooms: Discuss ways you might solve this problem without storing the full list of items seen.

Bloom filter: insert every element as it comes in

O(d)

~~positive~~ negative query → increase your count

Hash functions: use multiple hash functions
more distinct elements → more hashed values

(somewhat similar

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- Let $\mathbf{h} : U \to [0, 1]$ be a random hash function (with a real valued output)
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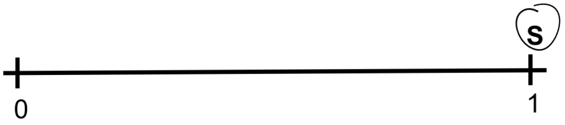

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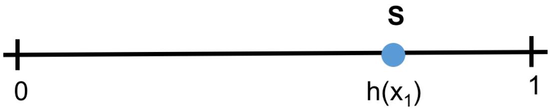

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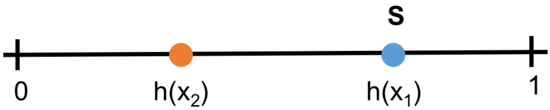

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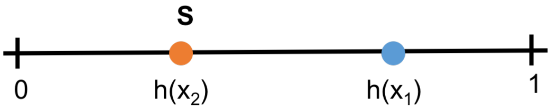

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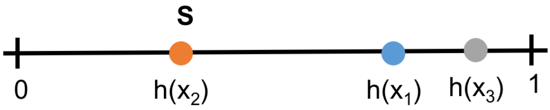

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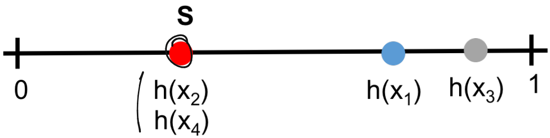
$x_2 = x_4$



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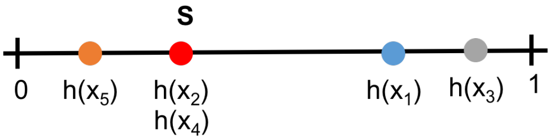

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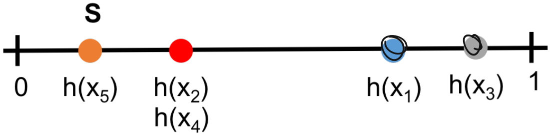
if # distinct items in large
s should be small



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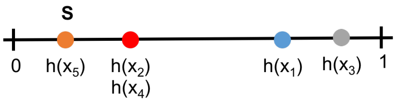

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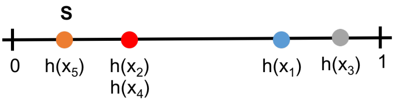

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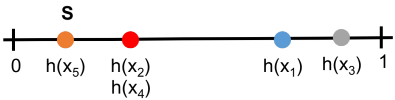

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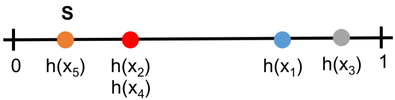

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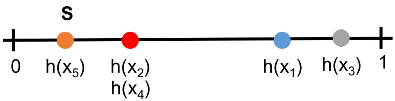

PERFORMANCE IN EXPECTATION

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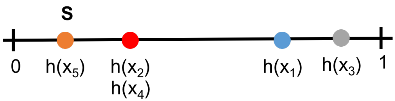

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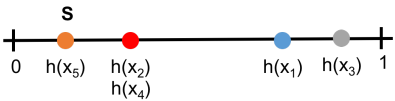

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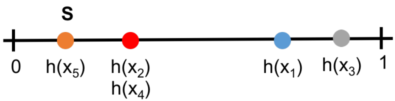

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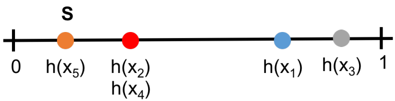

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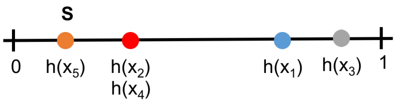

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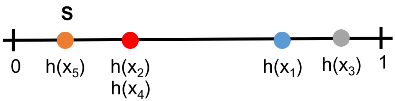

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- **Approximation is robust:** if $|\mathbf{s} - \mathbb{E}[\mathbf{s}]| \leq \epsilon \cdot \mathbb{E}[\mathbf{s}]$ for any $\epsilon \in (0, 1/2)$ and a small constant $c \leq 4$:

$$(1 - c\epsilon)d \leq \widehat{\mathbf{d}} \leq (1 + c\epsilon)d$$

So question is how well s concentrates around its mean.

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INITIAL CONCENTRATION BOUND

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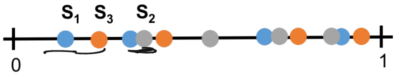

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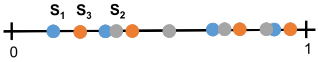
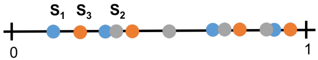
- For $i = 1, \ldots, n$
  - For j=1,..., k, $\mathbf{s}_j := \min(\mathbf{s}_j, \mathbf{h}_j(x_i))$
- $\mathbf{s} := \frac{1}{k}\sum_{j=1}^{k} \mathbf{s}_j$
- Return $\widehat{\mathbf{d}} = \frac{1}{\mathbf{s}} - 1$



- Setting $k = \frac{1}{\epsilon^2 \cdot \delta}$, algorithm returns $\widehat{\mathbf{d}}$ with $|d - \widehat{\mathbf{d}}| \leq 4\epsilon \cdot d$ with probability at least $1 - \delta$.

Hashing for Distinct Elements:

- Let $\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_k : U \to [0, 1]$ be random hash functions
- $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_k := 1$
- For $i = 1, \ldots, n$
  - For j=1,..., k, $\mathbf{s}_j := \min(\mathbf{s}_j, \mathbf{h}_j(x_i))$
- $\mathbf{s} := \frac{1}{k} \sum_{j=1}^{k} \mathbf{s}_j$
- Return $\widehat{\mathbf{d}} = \frac{1}{\mathbf{s}} - 1$

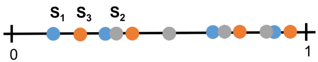


- Setting $k = \frac{1}{\epsilon^2 \cdot \delta}$, algorithm returns $\widehat{\mathbf{d}}$ with $|d - \widehat{\mathbf{d}}| \leq 4\epsilon \cdot d$ with probability at least $1 - \delta$.
- Space complexity is $k = \frac{1}{\epsilon^2 \cdot \delta}$ real numbers $s_1, \ldots, s_k$.

SPACE COMPLEXITY

Hashing for Distinct Elements:

- Let $\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_k : U \to [0, 1]$ be random hash functions
- $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_k := 1$
- For $i = 1, \ldots, n$
  - For j=1,..., k, $\mathbf{s}_j := \min(\mathbf{s}_j, \mathbf{h}_j(x_i))$
- $\mathbf{s} := \frac{1}{k} \sum_{j=1}^{k} \mathbf{s}_j$
- Return $\widehat{\mathbf{d}} = \frac{1}{\mathbf{s}} - 1$



x = y
h(x) = h(y)
w.p. 1

x ≠ y
h(x) = h(y)
with small prob.

- Setting $k = \frac{1}{\epsilon^2 \cdot \delta}$, algorithm returns $\widehat{\mathbf{d}}$ with $|d - \widehat{\mathbf{d}}| \leq 4\epsilon \cdot d$ with probability at least $1 - \delta$.

$\frac{1}{\delta} = 20$

- Space complexity is $k = \frac{1}{\epsilon^2 \cdot \delta}$ real numbers $s_1, \ldots, s_k$.
- $\delta = 5\%$ failure rate gives a factor 20 overhead in space complexity.