COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

Cameron Musco

University of Massachusetts Amherst. Fall 2020.

Lecture 3

By Thursday:

- Sign up for Piazza.
- Sign up for Gradescope (code on class website) and fill out the Gradescope consent poll on Piazza. Contact me via email if you don't consent to use Gradescope.

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First Problem Set: released Saturday, due 9/11 at 8pm in Gradescope.

- Remember you can complete in a group of up to 3 students, who all turn in one submission with three names on it.

WEEK 1 QUIZ

91 students completed the quizzes – make sure that if you are enrolled you are doing the quiz each week.

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Question 1: The expected number of inches of rain on Saturday is 2 and the expected number of inches on Sunday is 6. There is a 50% chance of rain on Saturday. If it rains on Saturday, there is a 75% chance of rain on Sunday. If it does not rain on Saturday, there is only a 25% chance of rain on Sunday. What is the expected number of inches of rainfall total over the weekend?

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Concerns: Probability/linear algebra background, proofs/derivations.

Last Class We Covered:

- Markov's inequality: the most fundamental concentration bound.
- Algorithmic applications of Markov's inequality, linearity of expectation, and indicator random variables:
    - Counting collisions to estimate CAPTCHA database size.
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- Markov's inequality: the most fundamental concentration bound.
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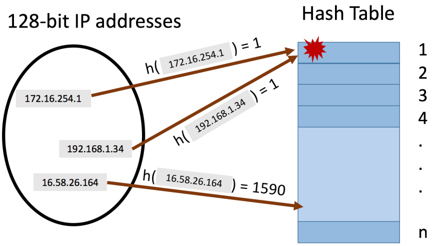
  - Counting collisions to estimate CAPTCHA database size.
  - Counting collisions to understand the runtime of hash tables with random hash functions.
- Collision counting is closely related to the birthday paradox.

Today:

- Finish up random hash functions and hash tables.
- See an applications of random hashing to load balancing in distributed systems.
- Through these applications learn about:
  - Chebyshev's inequality, which strengthens Markov's inequality.
  - The union bound, for understanding the probabilities of correlated random events.

We store m items from a large universe in a hash table with n positions.



- Want to show that when $\mathbf{h} : U \to [n]$ is a random hash function, query time is $O(1)$ with good probability.
- Equivalently: want to show that there are few collisions between hashed items.

When storing m items in a table of size n, the expected number of pairwise collisions (two items stored in the same slots) is:

$$\mathbb{E}[\mathbf{C}] = \frac{m(m-1)}{2n}.$$

m: total number of stored items, n: hash table size, $\mathbf{C}$: total pairwise collisions in table.

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- By Markov's inequality there no collisions with probability at least $\frac{7}{8}$.

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$O(1)$ query time, but we are using $O(m^2)$ space to store m items...

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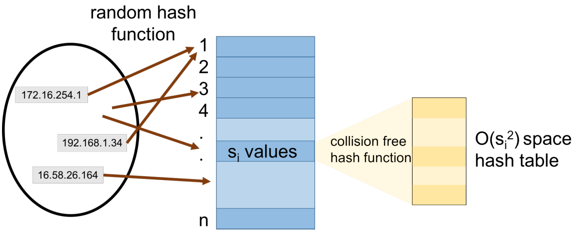

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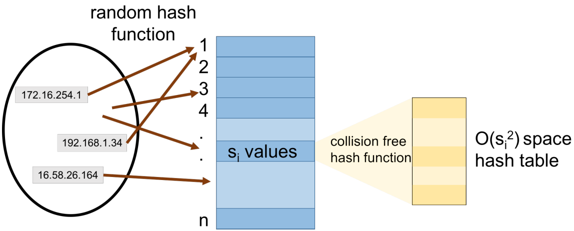

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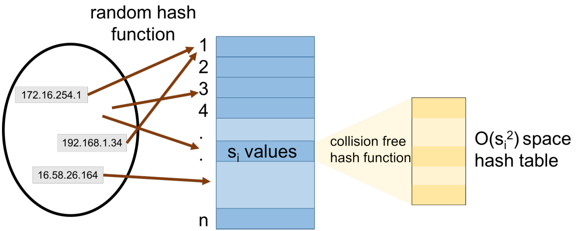

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- **Just Showed:** A random function is collision free with probability $\geq \frac{7}{8}$ so can just generate a random hash function and check if it is collision free.

SPACE USAGE

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Collisions again!

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$$= \frac{m}{n} + \frac{m(m-1)}{n^2} \leq 2 \text{ (If we set } n = m.\text{)}$$

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Near optimal space with $O(1)$ query time!

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So Far: we have assumed a **fully random hash function** $\mathbf{h}(x)$ with $\Pr[\mathbf{h}(x) = i] = \frac{1}{n}$ for $i \in 1, \ldots, n$ and $\mathbf{h}(x), \mathbf{h}(y)$ independent for $x \neq y$.

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- To compute a random hash function we have to store a table of x values and their hash values. Would take at least $O(m)$ space and $O(m)$ query time if we hash m values. Making our whole quest for $O(1)$ query time pointless!

x	h(x)
x_1	45
x_2	1004
x_3	10
⋮	⋮
x_m	12

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Efficient Alternative: Let p be a prime with $p \geq |U|$. Choose random $\mathbf{a}, \mathbf{b} \in [p]$ with $\mathbf{a} \neq 0$. Let:

$$\mathbf{h}(x) = (\mathbf{a}x + \mathbf{b} \mod p) \mod n.$$

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Remember: A fully random hash function is both 2-universal and pairwise independent. But it is not efficiently implementable.

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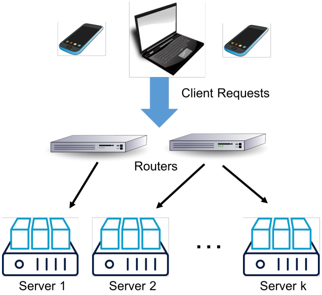
2. Then we'll show how a simple twist on Markov's can give a much stronger result.

Randomized Load Balancing:



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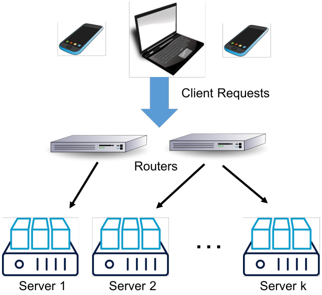

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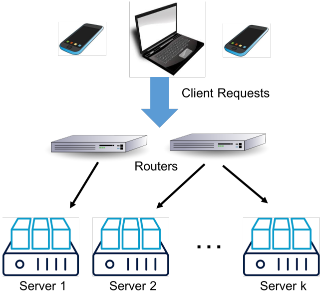

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- Often assignment is done via a random hash function. Why?

Expected Number of requests assigned to server i:

$$\mathbb{E}[R_i] = \sum_{j=1}^{n} \mathbb{E}[\mathbb{I}_{\text{request } j \text{ assigned to } i}] = \sum_{j=1}^{n} \Pr[j \text{ assigned to } i] = \frac{n}{k}.$$

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Not great...half the servers may be overloaded.

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