# COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

Prof. Cameron Musco

University of Massachusetts Amherst. Fall 2020.

Lecture 1

# MOTIVATION FOR THIS CLASS

People are increasingly interested in analyzing and learning from massive datasets.

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- Twitter receives 6,000 tweets per second, 500 million/day. Google receives 60,000 searches per second, 5.6 billion/day.
  - How do they process them to target advertisements? To predict trends? To improve their products?

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  - How do they process them to target advertisements? To predict trends? To improve their products?
- The Large Synoptic Survey Telescope will take high definition photographs of the sky, producing 15 terabytes of data/night.
  - How do they denoise and compress the images? How do they detect anomalies such as changing brightness or position of objects to alert researchers?

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# A NEW PARADIGM FOR ALGORITHM DESIGN

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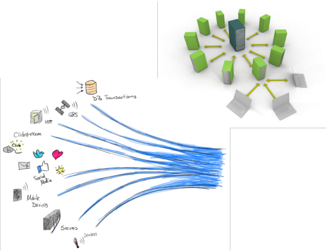
- Massive data sets require storage in a distributed manner or processing in a continuous stream.

vs.



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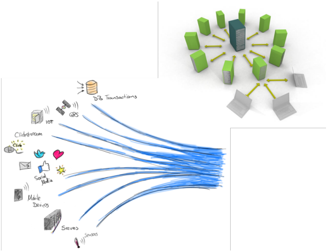


- Even ‘simple’ problems become very difficult in this setting.

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- How can Google estimate the number of unique search queries that are made in a given week? Given that no machine can store the full list of queries.
- When you use Shazam to identify a song from a recording, how does it provide an answer in < 10 seconds, without scanning over all $\sim$ 8 million audio files in its database.

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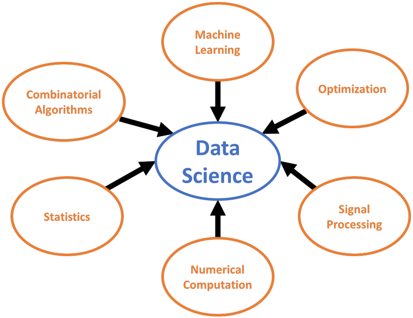

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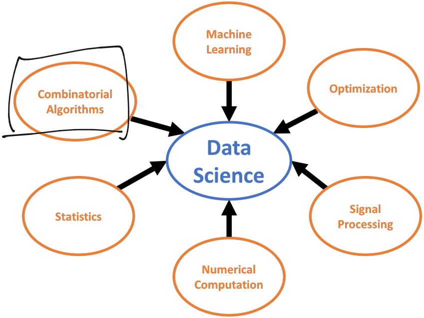


- Many techniques that aren't covered in the traditional CS algorithms curriculum.
- Emphasis on building comfort with mathematical tools that underly data science and machine learning.

# WHAT WE'LL COVER

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How can we efficiently compress large data sets in a way that lets us answer important algorithmic questions rapidly?

- Probability tools and concentration inequalities.
- Randomized hashing for efficient lookup, load balancing, and estimation. Bloom filters.
- Locality sensitive hashing and nearest neighbor search.
- Streaming algorithms: identifying frequent items in a data stream, counting distinct items, etc.
- Random compression of high-dimensional vectors: the Johnson-Lindenstrauss lemma, applications, and connections to the weirdness of high-dimensional geometry.

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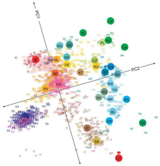
Section 2: Spectral Methods



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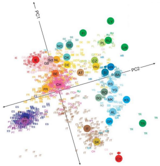

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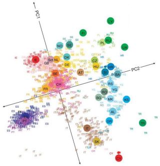

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- Principal component analysis, low-rank approximation, dimensionality reduction.
- The singular value decomposition (SVD) and its applications to PCA, low-rank approximation, LSI, MDS, …
- Spectral graph theory. Spectral clustering, community detection, network visualization.
- Computing the SVD on large datasets via iterative methods.

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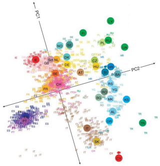

How do we identify the most important features of a dataset using linear algebraic techniques?

*If you open up the codes that are underneath [most data science applications] this is all linear algebra on arrays.*

– Michael Stonebraker

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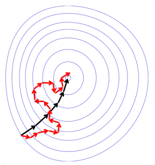
Section 3: Optimization



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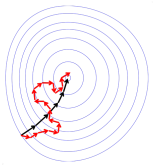

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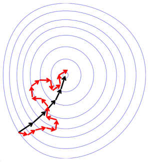

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- Gradient descent. Analysis for convex functions.
- Stochastic and online gradient descent.
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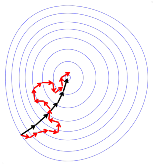

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A small taste of what you can find in COMPSCI 590OP or 690OP.

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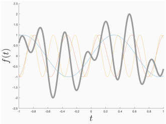
Section 4: Assorted Topics



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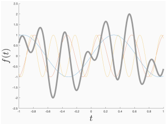


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- Discrete Fourier transform, fast Fourier transform.
- Differential privacy, algorithmic fairness.

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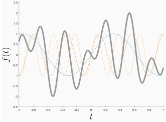


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Some flexibility here. Let me know what you are interested in!

# IMPORTANT TOPICS WE WON'T COVER

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- Systems/Software Tools.

hadoop TensorFlow

APACHE Spark™ Flink APACHE GIRAPH

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  - COMPSCI 589/689: Machine Learning

# STYLE OF THE COURSE

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**For example:** Baye's rule in conditional probability. What it means for a vector $x$ to be an eigenvector of a matrix $A$, orthogonal projection, greedy algorithms, divide-and-conquer algorithms.

See course webpage for logistics, policies, lecture notes, assignments, etc.:

http://people.cs.umass.edu/~cmusco/CS514F20/

See Moodle page for this link if you lose it and the lecture password.

**Professor:** Cameron Musco

- Email: cmusco@cs.umass.edu
- Office Hours: Tuesdays, 8am-9am and 2:15pm-3:15pm. See website for Zoom link (different than lecture)
- I encourage you to come as regularly as possible to ask questions/work together on practice problems.

**TAs:**

- Pratheba Selvaraju
- Shiv Shankar

See website for office hours, contact info, and Zoom links.

# PIAZZA AND PARTICIPATION

We will use Piazza for class discussion and questions.

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You may earn up to 5% extra credit for participation.

- Asking good clarifying questions and answering questions during the live lecture or on Piazza.
- Actively participating in office hours.
- Answering other students' or instructor questions on Piazza.
- Posting helpful/interesting links on Piazza, e.g., resources that cover class material, research articles related to the topics covered in class, etc.

- If you can, please turn on your video, but keep your mic muted when not asking a question.
- If you want to ask a question, either raise your hand, just unmute and interrupt me, and ask in chat.
- I encourage you to answer each others questions/discuss in chat during lecture.
- I will sometimes create random breakout rooms for you to work together/discuss problems. So if you are signed in to lecture please be ready to participate in these.

## TEXTBOOKS AND MATERIALS

We will use material from two textbooks (links to free online versions on the course webpage): *Foundations of Data Science* and *Mining of Massive Datasets*, but will follow neither closely.

- I will post optional readings a few days prior to each class.
- Lecture notes will be posted before each class, and annotated notes posted after class.

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- We strongly encourage working in groups, as it will make completing the problem sets much easier/more educational.
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- See Piazza for a thread to help you organize groups.

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Problem set submissions will be via Gradescope.

- See website for a link to join. **Entry Code: 9DV6G5**
- Since your emails, names, and grades will be stored in Gradescope we need your consent to use. See Piazza for a poll to give consent. Please complete by **next Thursday 9/3**.

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- Designed as a check-in that you are following the material, and to help me make adjustments as needed.
- Will take around 15-30 minutes per week, open notes.
- Will also include free response check-in questions to get your feedback on how the course is going, what material from the past week you find most confusing, interesting, etc.

Grade Breakdown:

- Problem Sets (6 total): 40%, weighted equally.
- Weekly Quizzes: 10%, weighted equally.
- Midterm (early October, take home): 25%.
- Final (early December, take home): 25%.
- Extra Credit: Up to 5% for participation, and lots more available on problem sets, for questions asked in class, etc.

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Academic Honesty:

- A first violation cheating on a homework, quiz, or other assignment will result in a 0 on that assignment.
- A second violation, or cheating on an exam will result in failing the class.

# DISABILITY SERVICES AND ACCOMODATIONS

UMass Amherst is committed to making reasonable, effective, and appropriate accommodations to meet the needs to students with disabilities.

- If you have a documented disability **on file with Disability Services**, you may be eligible for reasonable accommodations in this course.
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I understand that people have different learning needs, home situations, etc. If something isn't working for you in the class, please reach out and let's try to work it out.

Questions?

Get to know your classmates:

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- I’m going to split you into random breakout rooms.
- Come up with a list of as many things as you can that are common to all members of your group.
- They can’t be things that are common to say more than 50% of this class.
- Designate a scribe to write them down.
- The group with the largest list wins.

# Section 1: Randomized Methods & Sketching

# SOME PROBABILITY REVIEW

Consider a random X variable taking values in some finite set $S \subset \mathbb{R}$. E.g., for a random dice roll, $S = \{1, 2, 3, 4, 5, 6\}$.

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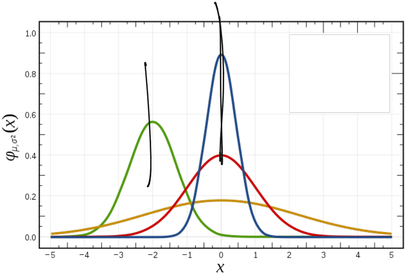
- **Expectation:** $\mathbb{E}[X] = \sum_{s \in S} \Pr(X = s) \cdot s.$



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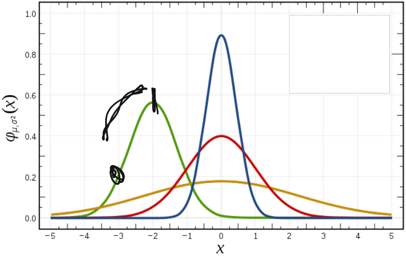
- **Variance:** $\text{Var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2].$



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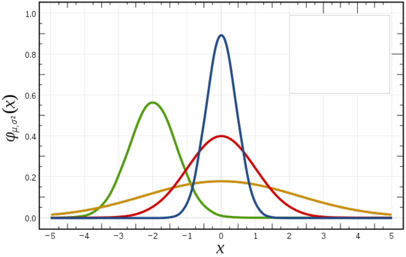


**Exercise:** Show that for any scalar $\alpha$, $\mathbb{E}[\alpha \cdot X] = \alpha \cdot \mathbb{E}[X]$ and $\text{Var}[\alpha \cdot X] = \alpha^2 \cdot \text{Var}[X]$.

Consider two random events $A$ and $B$.

$A \cap B$: event that both events $A$ *and* $B$ happen.

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- **Conditional Probability:**

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}.$$

A ∪ B

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- **Independence:** $A$ and $B$ are independent if:

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Using the definition of conditional probability, independence means:

$$\frac{\Pr(A \cap B)}{\Pr(B)} = \Pr(A) \implies \Pr(A \cap B) = \Pr(A) \cdot \Pr(B).$$

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For Example: What is the probability that for two independent dice rolls the first is a 6 and the second is odd?

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$$\Pr(D_1 = 6 \cap D_2 \in \{1,3,5\}) = \Pr(D_1 = 6) \cdot \Pr(D_2 \in \{1,3,5\})$$

$$= \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}$$

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**Independent Random Variables:** Two random variables X, Y are independent if for all $s, t$, $\mathrm{X} = s$ and $\mathrm{Y} = t$ are independent events. In other words:

$$\Pr(\mathrm{X} = s \cap \mathrm{Y} = t) = \Pr(\mathrm{X} = s) \cdot \Pr(\mathrm{Y} = t).$$

**Think-Pair-Share:** When are the expectation and variance linear?

I.e., under what conditions on $X$ and $Y$ do we have:

$$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

and

$$\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y].$$

$X, Y$: any two random variables.

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$ for any random variables $X$ and $Y$.

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Proof:

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**Proof:**

$$\mathbb{E}[X + Y] = \sum_{s \in S} \sum_{t \in T} \Pr(X = s \cap Y = t) \cdot (s + t)$$

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P(X=S)

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$$\begin{aligned}
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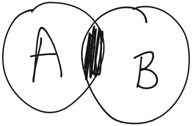
&= \sum_{s \in S} \sum_{t \in T} \Pr(X = s \cap Y = t) \cdot s + \sum_{s \in S} \sum_{t \in T} \Pr(X = s \cap Y = t) \cdot t \\
&= \sum_{s \in S} s \cdot \sum_{t \in T} \Pr(X = s \cap Y = t) + \sum_{t \in T} t \cdot \sum_{s \in S} \Pr(X = s \cap Y = t)
\end{aligned}$$

# LINEARITY OF EXPECTATION

$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$ for any random variables $X$ and $Y$.

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$$= \sum_{s \in S} s \cdot \Pr(X = s) + \sum_{t \in T} t \cdot \Pr(Y = t)$$

(law of total probability)

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**Proof:**

$$\begin{aligned}
\mathbb{E}[X+Y] &= \sum_{s\in S}\sum_{t\in T} \Pr(X = s \cap Y = t)\cdot(s+t) \\
&= \sum_{s\in S}\sum_{t\in T} \Pr(X = s \cap Y = t)\cdot s + \sum_{s\in S}\sum_{t\in T} \Pr(X = s \cap Y = t)\cdot t \\
&= \sum_{s\in S} s \cdot \sum_{t\in T} \Pr(X = s \cap Y = t) + \sum_{t\in T} t \cdot \sum_{s\in S} \Pr(X = s \cap Y = t) \\
&= \sum_{s\in S} s \cdot \Pr(X = s) + \sum_{t\in T} t \cdot \Pr(Y = t) \\
&\qquad\qquad \text{(law of total probability)} \\
&= \mathbb{E}[X] + \mathbb{E}[Y].
\end{aligned}$$

$$\mathrm{Var}[X + Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$$

$\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y]$ when $X$ and $Y$ are independent.

$\text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y]$ when X and Y are independent.

**Claim 1: (exercise)** $\text{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$

$\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y]$ when $X$ and $Y$ are independent.

**Claim 1: (exercise)** $\text{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$ (via linearity of expectation)

$\text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y]$ when $X$ and $Y$ are independent.

**Claim 1: (exercise)** $\text{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$ (via linearity of expectation)

**Claim 2: (exercise)** $\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$ when $X, Y$ are independent.

$\text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y]$ when $X$ and $Y$ are independent.

**Claim 1: (exercise)** $\text{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$ (via linearity of expectation)

**Claim 2: (exercise)** $\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$ when $X, Y$ are independent.

**Together give:**

$\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y]$ when $X$ and $Y$ are independent.

**Claim 1: (exercise)** $\text{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$ (via linearity of expectation)

**Claim 2: (exercise)** $\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$ when $X, Y$ are independent.

**Together give:**

$$\text{Var}[X+Y] = \mathbb{E}[(X+Y)^2] - \mathbb{E}[X+Y]^2$$

$\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y]$ when $X$ and $Y$ are independent.

**Claim 1: (exercise)** $\text{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$ (via linearity of expectation)

**Claim 2: (exercise)** $\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$ when $X, Y$ are independent.

**Together give:**

$$\begin{aligned}\text{Var}[X+Y] &= \mathbb{E}[(X+Y)^2] - \mathbb{E}[X+Y]^2 \\ &= \mathbb{E}[X^2] + 2\mathbb{E}[XY] + \mathbb{E}[Y^2] - (\mathbb{E}[X] + \mathbb{E}[Y])^2\end{aligned}$$

(linearity of expectation)

$\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y]$ when $X$ and $Y$ are independent.

**Claim 1: (exercise)** $\text{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$ (via linearity of expectation)

**Claim 2: (exercise)** $\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$ when $X, Y$ are independent.

**Together give:**

$$\begin{aligned} \text{Var}[X+Y] &= \mathbb{E}[(X+Y)^2] - \mathbb{E}[X+Y]^2 \\ &= \mathbb{E}[X^2] + 2\mathbb{E}[XY] + \mathbb{E}[Y^2] - (\mathbb{E}[X] + \mathbb{E}[Y])^2 \\ &\qquad \text{(linearity of expectation)} \\ &= \mathbb{E}[X^2] + 2\mathbb{E}[XY] + \mathbb{E}[Y^2] - \mathbb{E}[X]^2 - 2\mathbb{E}[X] \cdot \mathbb{E}[Y] - \mathbb{E}[Y]^2 \end{aligned}$$

$\text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y]$ when $X$ and $Y$ are independent.

**Claim 1: (exercise)** $\text{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$ (via linearity of expectation)

**Claim 2: (exercise)** $\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$ when $X, Y$ are independent.

**Together give:**

$$\begin{aligned}
\text{Var}[X + Y] &= \mathbb{E}[(X + Y)^2] - \mathbb{E}[X + Y]^2 \\
&= \mathbb{E}[X^2] + 2\mathbb{E}[XY] + \mathbb{E}[Y^2] - (\mathbb{E}[X] + \mathbb{E}[Y])^2 \\
&\qquad\qquad\qquad \text{(linearity of expectation)} \\
&= \mathbb{E}[X^2] + 2\mathbb{E}[XY] + \mathbb{E}[Y^2] - \mathbb{E}[X]^2 - 2\mathbb{E}[X] \cdot \mathbb{E}[Y] - \mathbb{E}[Y]^2
\end{aligned}$$

$\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y]$ when $X$ and $Y$ are independent.

**Claim 1: (exercise)** $\text{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$ (via linearity of expectation)

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**Together give:**

$$\begin{aligned} \text{Var}[X+Y] &= \mathbb{E}[(X+Y)^2] - \mathbb{E}[X+Y]^2 \\ &= \mathbb{E}[X^2] + 2\mathbb{E}[XY] + \mathbb{E}[Y^2] - (\mathbb{E}[X] + \mathbb{E}[Y])^2 \\ &\qquad \text{(linearity of expectation)} \\ &= \mathbb{E}[X^2] + 2\mathbb{E}[XY] + \mathbb{E}[Y^2] - \mathbb{E}[X]^2 - 2\mathbb{E}[X] \cdot \mathbb{E}[Y] - \mathbb{E}[Y]^2 \\ &= \mathbb{E}[X^2] + \mathbb{E}[Y^2] - \mathbb{E}[X]^2 - \mathbb{E}[Y]^2 \end{aligned}$$

$\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y]$ when $X$ and $Y$ are independent.

**Claim 1: (exercise)** $\text{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$ (via linearity of expectation)

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**Together give:**

$$\begin{aligned}
\text{Var}[X+Y] &= \mathbb{E}[(X+Y)^2] - \mathbb{E}[X+Y]^2 \\
&= \mathbb{E}[X^2] + 2\mathbb{E}[XY] + \mathbb{E}[Y^2] - (\mathbb{E}[X] + \mathbb{E}[Y])^2 \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(linearity of expectation)} \\
&= \mathbb{E}[X^2] + 2\mathbb{E}[XY] + \mathbb{E}[Y^2] - \mathbb{E}[X]^2 - 2\mathbb{E}[X] \cdot \mathbb{E}[Y] - \mathbb{E}[Y]^2 \\
&= \mathbb{E}[X^2] + \mathbb{E}[Y^2] - \mathbb{E}[X]^2 - \mathbb{E}[Y]^2 \\
&= \text{Var}[X] + \text{Var}[Y].
\end{aligned}$$