COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

Cameron Musco

University of Massachusetts Amherst. Fall 2019.

Lecture 9

LOGISTICS

- Problem Set 2 was released on 9/28. **Due Friday 10/11**.
- Problem Set 1 should be graded by the end of this week.
- Midterm on Thursday 10/17. Will cover material through this week, but not material next week (10/8 and 10/10).

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- Midterm on Thursday 10/17. Will cover material through this week, but not material next week (10/8 and 10/10).
- This Thursday, will have a MAP (Midterm Assessment Process).
    - Someone from the Center for Teaching & Learning will collect feedback from you during the first 20 minutes of class.
    - Will be summarized and relayed to me anonymously, so I can make any adjustments and incorporate suggestions to help you learn the material better.

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- Deterministic algorithms: Boyer-Moore majority algorithm and Misra-Gries summaries.
- Randomized algorithm: Count-Min sketch
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This Class: Randomized dimensionality reduction.

- The extremely powerful Johnson-Lindenstrauss Lemma and random projection.
- Linear algebra warm up.

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- The human genome contains 3 billion+ base pairs. Genetic datasets often contain information on 100s of thousands+ mutations and genetic markers.

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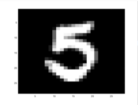

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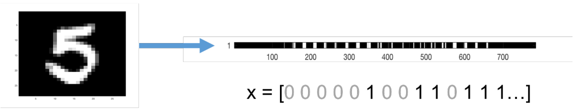

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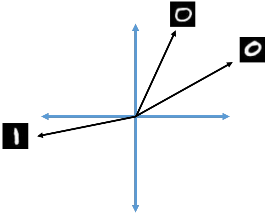

Similarities/distance between vectors (e.g., $\langle x, y \rangle$, $\|x - y\|_2$) have meaning for underlying datapoints.

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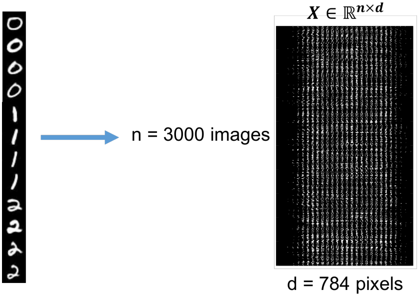

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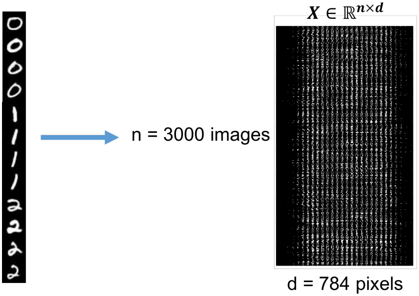

Many data points $n \implies$ tall. Many dimensions $d \implies$ wide.

Dimensionality Reduction: Compress data points so that they lie in many fewer dimensions.

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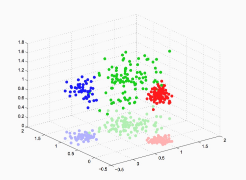

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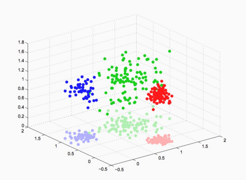

Generally will not consider directly how well $\tilde{x}_i$ approximates x_i.

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- Principal component analysis
- Latent semantic analysis (LSA)

Raw Text

Term Document Representation

x_1 = [0 0 0 1 0 0 1 1 0 1 1 0 1…]
x_2 = [0 1 1 1 0 1 0 1 1 1 0 1 0…]
…
x_n = [1 0 1 0 1 0 0 1 1 0 1 0 0…]

Latent Representation

$\tilde{x}_1$ = [1.1 2.4 0 -5]
$\tilde{x}_2$= [-1.4 5.6 7 -.1]
…
$\tilde{x}_n$= [10.6 -1 -1 2.2]

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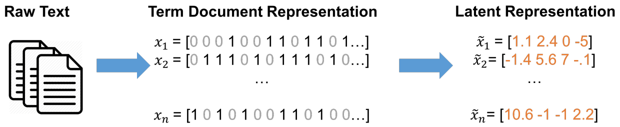

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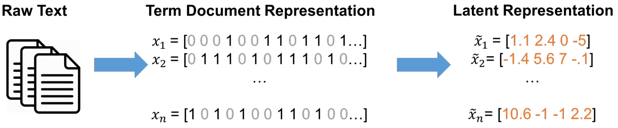

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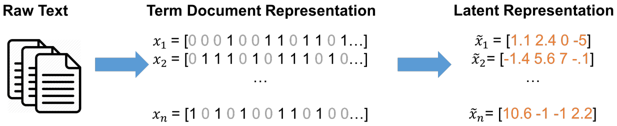

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Compressing data makes it more efficient to work with. May also remove extraneous information/noise.

Low Distortion Embedding: Given $x_1, \ldots, x_n \in \mathbb{R}^d$, distance function D, and error parameter $\epsilon \geq 0$, find $\tilde{x}_1, \ldots \tilde{x}_n \in \mathbb{R}^{d'}$ (where $d' \ll d$) and distance function $\tilde{D}$ such that for all $i, j \in [n]$:

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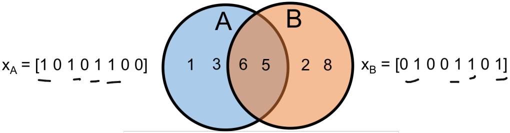

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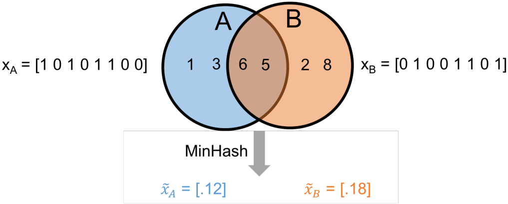

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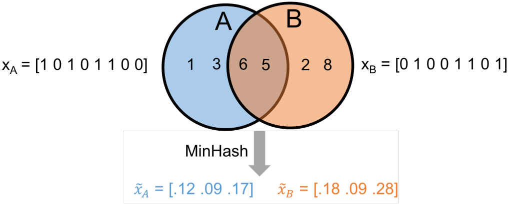

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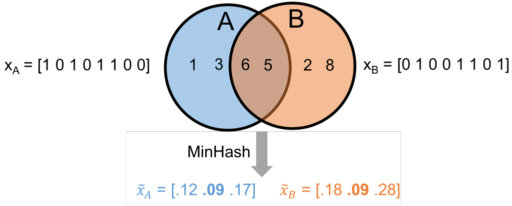

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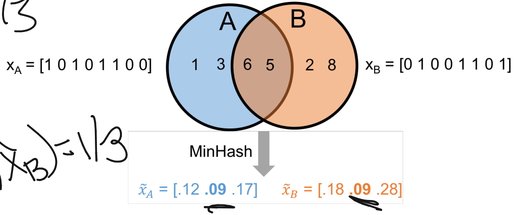

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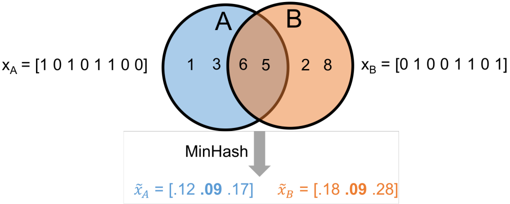

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- Reduce dimension from $d = |U|$ to r.

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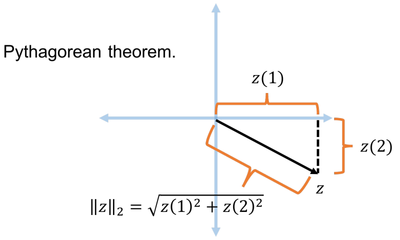
Recall that for $z \in \mathbb{R}^m$, $\|z\|_2 = \sqrt{\sum_{i=1}^m z(i)^2}$.



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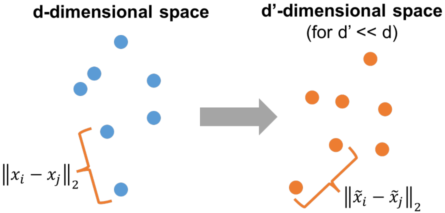

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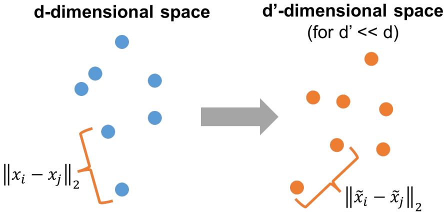

Can use $\tilde{x}_1, \ldots, \tilde{x}_n$ in place of $x_1, \ldots, x_n$ in many applications: clustering, SVM, near neighbor search, etc.

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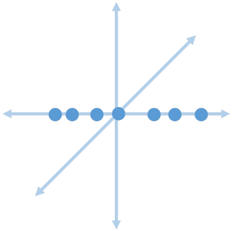

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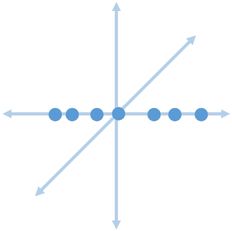

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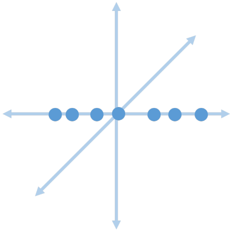

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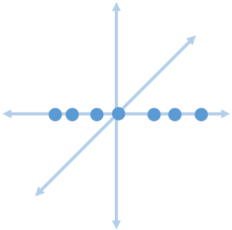

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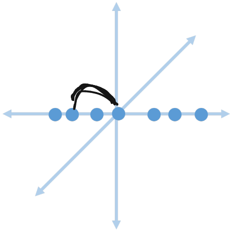

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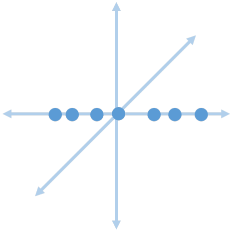

Set $d' = 1$ and $\tilde{x}_i = x_i(1)$ (i.e., $\tilde{x}_i$ is just a single number.).

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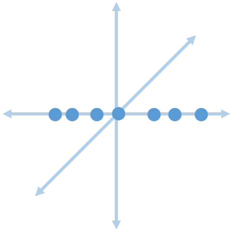

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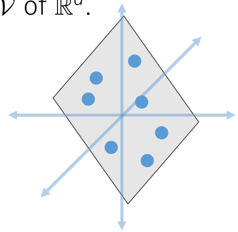

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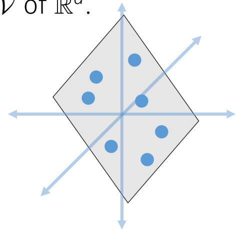

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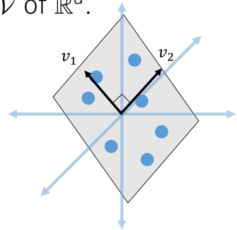

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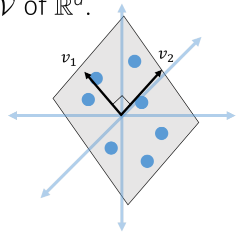

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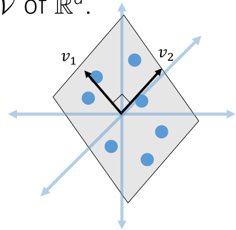

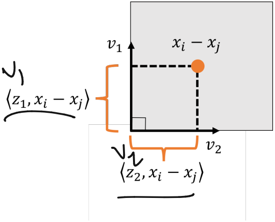

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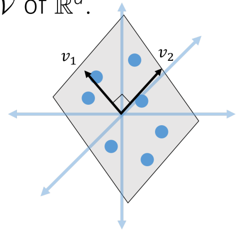

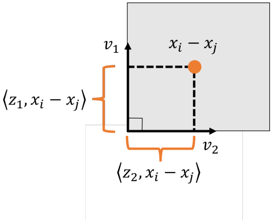

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- An embedding with no distortion from any d into $d' = k$.
- $\mathbf{V}^T : \mathbb{R}^d \to \mathbb{R}^k$ is a linear map giving our dimension reduction.

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Very surprising! Powerful result with a simple (naive) construction: applying a random linear transformation to a set of points preserves the distances between all those points with high probability.

For any $x_1, \ldots x_n$, and $\mathbf{\Pi} \in \mathbb{R}^{d \times d'}$ chosen with each entry chosen i.i.d. as $\frac{1}{\sqrt{d'}} \cdot \mathcal{N}(0,1)$, with high probability, letting $\tilde{x}_i = \mathbf{\Pi} x_i$:

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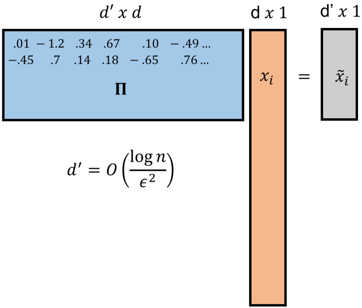

RANDOM PROJECTION

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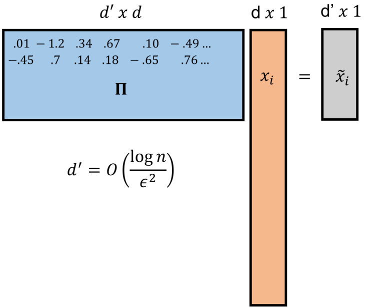

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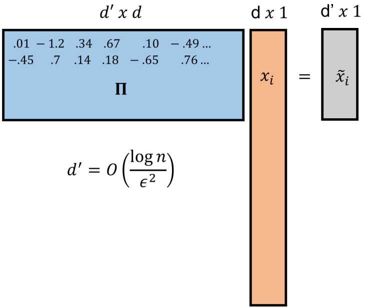

- $\mathbf{\Pi}$ is known as a random projection.
- Data oblivious transformation. Stark contrast to methods like PCA.

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- Compression can also be easily performed in parallel on different servers.
- When new data points are added, can be easily compressed, without updating existing points.

CONNECTION TO SIMHASH

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CONNECTION TO SIMHASH

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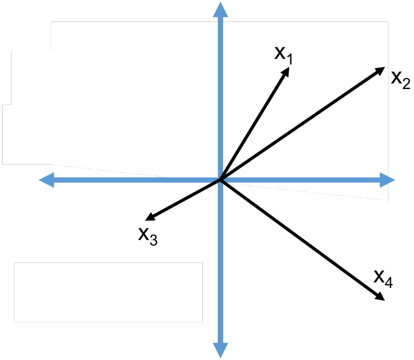

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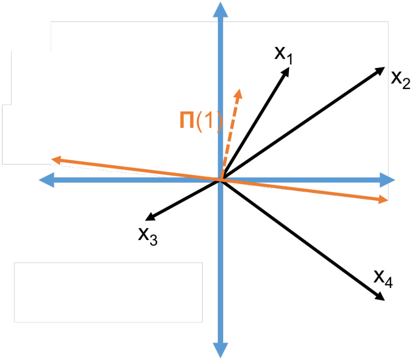

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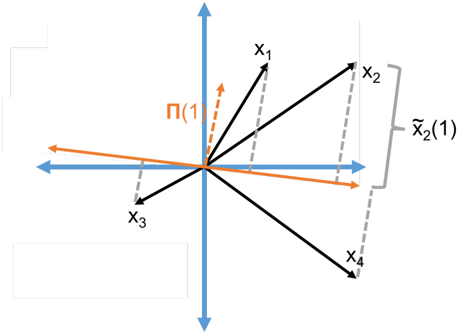

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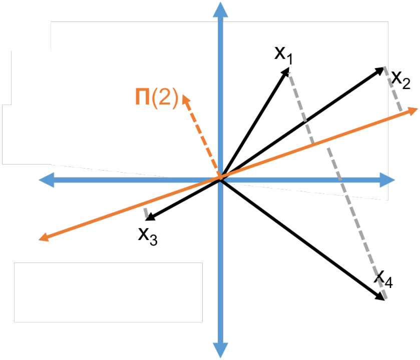

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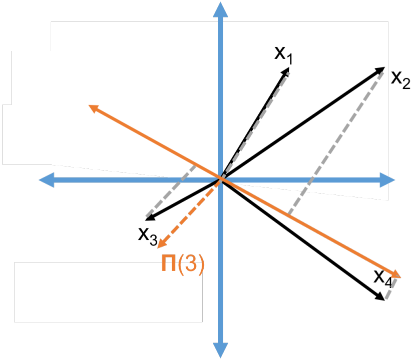

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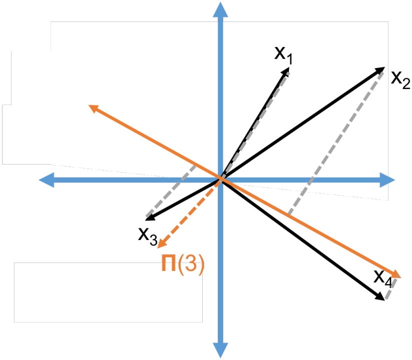

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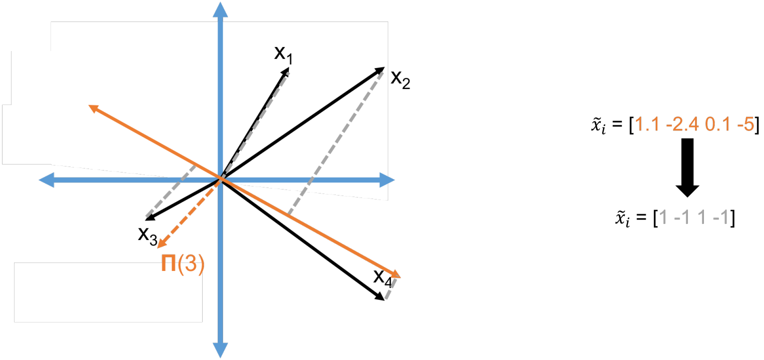

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Can be proven from first principles. Will see next.

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DISTRIBUTIONAL JL $\implies$ JL

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Setting δ to any fixed constant yields the JL lemma.

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DISTRIBUTIONAL JL PROOF

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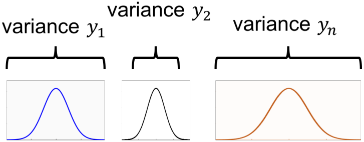

$$\tilde{\mathbf{y}}(j) = \frac{1}{\sqrt{m}}(\boldsymbol{g}_1 \cdot y_1 + \boldsymbol{g}_2 \cdot y_2 + \ldots + \boldsymbol{g}_n \cdot y_n)$$

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Stability of Gaussian Random Variables. For independent $a \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $b \sim \mathcal{N}(\mu_2, \sigma_2^2)$ we have:

$$a + b \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

Thus, $\tilde{\mathbf{y}}(j) \sim \frac{1}{\sqrt{m}} \mathcal{N}(0, \|y\|_2^2)$. *Rotational invariance of the Gaussian distribution.*

Stability is another explanation for the central limit theorem.

So far: Letting $\mathbf{\Pi} \in \mathbb{R}^{d \times m}$ have each entry chosen i.i.d. as $\frac{1}{\sqrt{m}} \cdot \mathcal{N}(0,1)$, for any $y \in \mathbb{R}^d$, letting $\tilde{\mathbf{y}} = \mathbf{\Pi} y$:

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How is $\|\tilde{\mathbf{y}}\|_2^2$ distributed? Does it concentrate?

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Lemma: (Chi-Squared Concentration) Letting $\mathbf{Z}$ be a Chi-Squared random variable with m degrees of freedom,

$$\Pr\left[|\mathbf{Z} - \mathbb{E}\mathbf{Z}| \geq \epsilon \mathbb{E}\mathbf{Z}\right] \leq 2e^{-m\epsilon^2/8}.$$

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If we set $m = O\left(\frac{\log(1/\delta)}{\epsilon^2}\right)$, with probability $1 - O(e^{-\log(1/\delta)}) \geq 1 - \delta$:

$$(1-\epsilon)\|y\|_2^2 \leq \|\tilde{\mathbf{y}}\|_2^2 \leq (1+\epsilon)\|y\|_2^2.$$

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Gives the distributional JL Lemma and thus the classic JL Lemma.

Questions?