COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

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University of Massachusetts Amherst. Fall 2019.
Lecture 3

By Thursday:

- Sign up for Piazza.
- Pick a problem set group with 3 people and have one member email me the names of the members and a group name.
- Fill out the Gradescope consent poll on Piazza and contact me via email if you don't consent.
- The first problem set will be **available 9/12 and due 9/26.**

Materials:

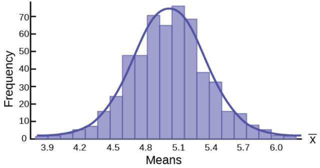
- Lectures are being recorded via Echo360 and on posted in Moodle. Link on class webpage.
- Slides will be available before class.

Last Class We Covered:

- Random hash functions, collision free hashing, and two-level hashing (analysis with linearity of expectation and Markov's inequality.)
- 2-universal and pairwise independent hash functions.
- Chebyshev's inequality and the law of large numbers.
- Application to randomized load balancing.

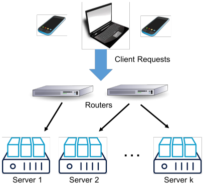
Today: We'll see even stronger concentration bounds than Chebyshev's inequality – exponential tail bounds.

- Will show a version of the central limit theorem.



First: We'll show learn about the union bound and apply it to randomized load balancing.

Randomized Load Balancing:



- n requests randomly assigned to k servers.
- Letting $\mathbf{R}_i$ be the number of requests assigned to server i, $\mathbb{E}[\mathbf{R}_i] = \frac{n}{k}$ and $\text{Var}[\mathbf{R}_i] \leq \frac{n}{k}$.
- By Chebyshev's inequality: $\Pr\left(\mathbf{R}_i \geq \frac{2n}{k}\right) \leq \frac{\text{Var}[\mathbf{R}_i]}{(n/k)^2} = \frac{k}{n}$.
- Also applies when assignment is with a pairwise independent hash function (a good exercise to work through).

What is the probability that the maximum server load exceeds $2 \cdot \mathbb{E}[\mathbf{R}_i] = \frac{2n}{k}$. I.e., that some server is overloaded if we give each $\frac{2n}{k}$ capacity?

$$\Pr\left(\max_i(\mathbf{R}_i) \geq \frac{2n}{k}\right) = \Pr\left(\left[\mathbf{R}_1 \geq \frac{2n}{k}\right] \cup \left[\mathbf{R}_2 \geq \frac{2n}{k}\right] \cup \ldots \cup \left[\mathbf{R}_k \geq \frac{2n}{k}\right]\right)$$

n: total number of requests, k: number of servers randomly assigned requests, $\mathbf{R}_i$: number of requests assigned to server i. $\mathbb{E}[\mathbf{R}_i] = \frac{n}{k}$. $\text{Var}[\mathbf{R}_i] = \frac{n}{k}$.

What is the probability that the maximum server load exceeds $2 \cdot \mathbb{E}[R_i] = \frac{2n}{k}$. I.e., that some server is overloaded if we give each $\frac{2n}{k}$ capacity?

$$\Pr\left(\max_i(R_i) \geq \frac{2n}{k}\right) = \Pr\left(\left[R_1 \geq \frac{2n}{k}\right] \text{ or } \left[R_2 \geq \frac{2n}{k}\right] \text{ or } \ldots \text{ or } \left[R_k \geq \frac{2n}{k}\right]\right)$$

n: total number of requests, k: number of servers randomly assigned requests, R_i: number of requests assigned to server i. $\mathbb{E}[R_i] = \frac{n}{k}$. $\text{Var}[R_i] = \frac{n}{k}$.

What is the probability that the maximum server load exceeds $2 \cdot \mathbb{E}[\mathsf{R}_i] = \frac{2n}{k}$. I.e., that some server is overloaded if we give each $\frac{2n}{k}$ capacity?

$$\Pr\left(\max_i(\mathsf{R}_i) \geq \frac{2n}{k}\right) = \Pr\left(\bigcup_{i=1}^{k}\left[\mathsf{R}_i \geq \frac{2n}{k}\right]\right)$$

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What is the probability that the maximum server load exceeds $2 \cdot \mathbb{E}[\mathbf{R}_i] = \frac{2n}{k}$. I.e., that some server is overloaded if we give each $\frac{2n}{k}$ capacity?

$$\Pr\left(\max_i(\mathbf{R}_i) \geq \frac{2n}{k}\right) = \Pr\left(\bigcup_{i=1}^{k}\left[\mathbf{R}_i \geq \frac{2n}{k}\right]\right)$$

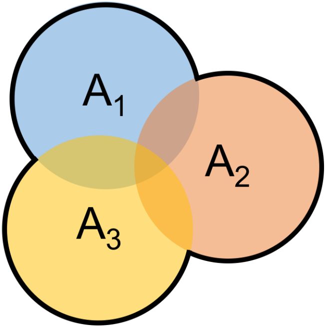
We want to show that $\Pr\left(\bigcup_{i=1}^{k}\left[\mathbf{R}_i \geq \frac{2n}{k}\right]\right)$ is small.

How do we do this? Note that $\mathbf{R}_1, \ldots, \mathbf{R}_k$ are correlated in a somewhat complex way.

n: total number of requests, k: number of servers randomly assigned requests, $\mathbf{R}_i$: number of requests assigned to server i. $\mathbb{E}[\mathbf{R}_i] = \frac{n}{k}$. $\mathrm{Var}[\mathbf{R}_i] = \frac{n}{k}$.

Union Bound: For any random events $A_1, A_2, ..., A_k$,

$$\Pr\left(A_1 \cup A_2 \cup \ldots \cup A_k\right) \leq \Pr(A_1) + \Pr(A_2) + \ldots + Pr(A_k).$$



When is the union bound tight? When $A_1, ..., A_k$ are all disjoint.

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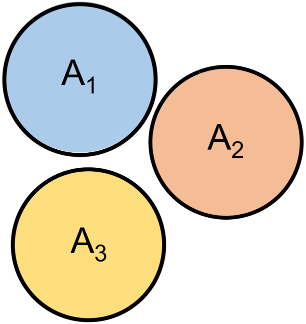

When is the union bound tight? When $A_1, ..., A_k$ are all disjoint.

What is the probability that the maximum server load exceeds $2 \cdot \mathbb{E}[\mathbf{R}_i] = \frac{2n}{k}$. I.e., that some server is overloaded if we give each $\frac{2n}{k}$ capacity?

$$\Pr\left(\max_i(\mathbf{R}_i) \geq \frac{2n}{k}\right) = \Pr\left(\bigcup_{i=1}^{k}\left[\mathbf{R}_i \geq \frac{2n}{k}\right]\right)$$

$$\leq \sum_{i=1}^{k} \Pr\left(\left[\mathbf{R}_i \geq \frac{2n}{k}\right]\right)$$

$$\leq \sum_{i=1}^{k} \frac{k}{n} = \frac{k^2}{n} \qquad \text{(Union Bound)}$$

As long as $k \leq O(\sqrt{n})$, with good probability, the maximum server load will be small (compared to the expected load).

n: total number of requests, k: number of servers randomly assigned requests, $\mathbf{R}_i$: number of requests assigned to server i. $\mathbb{E}[\mathbf{R}_i] = \frac{n}{k}$. $\mathrm{Var}[\mathbf{R}_i] = \frac{n}{k}$.

The number of servers must be small compared to the number of requests ($k = O(\sqrt{n})$) for the maximum load to be bounded in comparison to the expected load with good probability.

- There are many requests routed to a relatively small number of servers so the load seen on each server is close to what is expected via law of large numbers.
- **A Useful Exercise:** Given n requests, and assuming all servers have fixed capacity C, how many servers should you provision so that with probability $\geq 99/100$ no server is assigned more than C requests?

n: total number of requests, k: number of servers randomly assigned requests.

Questions on union bound, Chebyshev's inequality, random hashing?

We flip $n = 100$ independent coins, each are heads with probability 1/2 and tails with probability 1/2. Let H be the number of heads.

$$\mathbb{E}[\mathsf{H}] = \frac{n}{2} = 50 \text{ and } \mathrm{Var}[\mathsf{H}] = \frac{n}{4} = 25 \to s.d. = 5$$

Markov's:	Chebyshev's:	In Reality:
$\Pr(\mathsf{H} \geq 60) \leq .833$	$\Pr(\mathsf{H} \geq 60) \leq .25$	$\Pr(\mathsf{H} \geq 60) = 0.0284$
$\Pr(\mathsf{H} \geq 70) \leq .714$	$\Pr(\mathsf{H} \geq 70) \leq .0625$	$\Pr(\mathsf{H} \geq 70) = .000039$
$\Pr(\mathsf{H} \geq 80) \leq .625$	$\Pr(\mathsf{H} \geq 80) \leq .04$	$\Pr(\mathsf{H} \geq 80) < 10^{-9}$

H has a simple Binomial distribution, so can compute these probabilities exactly.

To be fair.... Markov and Chebyshev's inequalities apply much more generally than to Binomial random variables like coin flips.

Can we obtain tighter concentration bounds that still apply to very general distributions?

- Markov's: $\Pr(X \geq t) \leq \frac{\mathbb{E}[X]}{t}$. First Moment.
- Chebyshev's: $\Pr(|X - \mathbb{E}[X]| \geq t) = \Pr(|X - \mathbb{E}[X]|^2 \geq t^2) \leq \frac{\mathrm{Var}[X]}{t^2}$. Second Moment.
- What if we just apply Markov's inequality to even higher moments?

Consider any random variable X:

$$\Pr(|X - \mathbb{E}[X]| \geq t) = \Pr\left((X - \mathbb{E}[X])^4 \geq t^4\right) \leq \frac{\mathbb{E}\left[(X - \mathbb{E}[X])^4\right]}{t^4}.$$

Application to Coin Flips: Recall: $n = 100$ independent fair coins, H is the number of heads.

- Bound the fourth moment:

$$\mathbb{E}\left[(H - \mathbb{E}[H])^4\right] = \mathbb{E}\left[\left(\sum_{i=1}^{100} H_i - 50\right)^4\right] = \sum_{i,j,k,\ell} c_{ijk\ell}\mathbb{E}[H_i H_j H_k H_\ell] = 1862.5$$

  where $H_i = 1$ if coin flip i is heads and 0 otherwise. Then apply some messy calculations...
- Apply Fourth Moment Bound: $\Pr\left(|H - \mathbb{E}[H]| \geq t\right) \leq \frac{1862.5}{t^4}$.

Chebyshev's:	4^{th} Moment:	In Reality:
$\Pr(\mathbf{H} \geq 60) \leq .25$	$\Pr(\mathbf{H} \geq 60) \leq .186$	$\Pr(\mathbf{H} \geq 60) = 0.0284$
$\Pr(\mathbf{H} \geq 70) \leq .0625$	$\Pr(\mathbf{H} \geq 70) \leq .0116$	$\Pr(\mathbf{H} \geq 70) = .000039$
$\Pr(\mathbf{H} \geq 80) \leq .04$	$\Pr(\mathbf{H} \geq 80) \leq .0023$	$\Pr(\mathbf{H} \geq 80) < 10^{-9}$

Can we just keep applying Markov's inequality to higher and higher moments and getting tighter bounds?

- Yes! To a point.
- In fact – don't need to just apply Markov's to $|X - \mathbb{E}[X]|^k$ for some k. Can apply to any monotonic function $f(|X - \mathbb{E}[X]|)$.
- Why monotonic? $\Pr(|X - \mathbb{E}[X]| > t) = \Pr(f(|X - \mathbb{E}[X]|) > f(t))$.

$\mathbf{H}$: total number heads in 100 random coin flips. $\mathbb{E}[\mathbf{H}] = 50$.

Moment Generating Function: Consider for any $t > 0$:

$$M_t(X) = e^{t\cdot(X-\mathbb{E}[X])} = \sum_{k=0}^{\infty} \frac{t^k (X-\mathbb{E}[X])^k}{k!}$$

- $M_t(X)$ is monotonic for any $t > 0$.
- Weighted sum of all moments, with t controlling how slowly the weights fall off (larger t = slower falloff).
- Choosing t appropriately gives a number of very powerful exponential concentration bounds (exponential tail bounds).
- Chernoff bound, Bernstein inequalities, Hoeffding's inequality, Azuma's inequality, Berry-Esseen theorem, etc.

Bernstein Inequality: Consider independent random variables $X_1, \ldots, X_n$ all falling in $[-M, M]$. Let $\mu = \mathbb{E}[\sum_{i=1}^n X_i]$ and $\sigma^2 = \mathrm{Var}[\sum_{i=1}^n X_i] = \sum_{i=1}^n \mathrm{Var}[X_i]$. For any $t \geq 0$:

$$\Pr\left(\left|\sum_{i=1}^n X_i - \mu\right| \geq t\right) \leq 2\exp\left(-\frac{t^2}{2\sigma^2 + \frac{4}{3}Mt}\right).$$

Bernstein Inequality: Consider independent random variables $X_1, \ldots, X_n$ all falling in [-1,1]. Let $\mu = \mathbb{E}[\sum_{i=1}^n X_i]$ and $\sigma^2 = \mathrm{Var}[\sum_{i=1}^n X_i] = \sum_{i=1}^n \mathrm{Var}[X_i]$. For any $s \geq 0$:

$$\Pr\left(\left|\sum_{i=1}^n X_i - \mu\right| \geq s\sigma\right) \leq 2\exp\left(-\frac{s^2}{4}\right).$$

Assume that $M = 1$ and plug in $t = s \cdot \sigma$ for $s \leq \sigma$.

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Assume that $M = 1$ and plug in $t = s \cdot \sigma$ for $s \leq \sigma$.

Compare to Chebyshev's: $\Pr\left(\left|\sum_{i=1}^n X_i - \mu\right| \geq s\sigma\right) \leq \frac{1}{s^2}$.

- An exponentially stronger dependence on s!

Consider again bounding the number of heads H in $n = 100$ independent coin flips.

Chebyshev's:	Bernstein:	In Reality:
$\Pr(\mathsf{H} \geq 60) \leq .25$	$\Pr(\mathsf{H} \geq 60) \leq .15$	$\Pr(\mathsf{H} \geq 60) = 0.0284$
$\Pr(\mathsf{H} \geq 70) \leq .0625$	$\Pr(\mathsf{H} \geq 70) \leq .00086$	$\Pr(\mathsf{H} \geq 70) = .000039$
$\Pr(\mathsf{H} \geq 80) \leq .04$	$\Pr(\mathsf{H} \geq 80) \leq 3^{-7}$	$\Pr(\mathsf{H} \geq 80) < 10^{-9}$

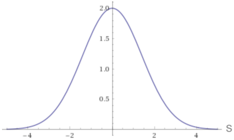
Getting much closer to the true probability.

H: total number heads H in 100 random coin flips. $\mathbb{E}[\mathsf{H}] = 50$.

Bernstein Inequality: Consider independent random variables $X_1, \ldots, X_n$ falling in [-1,1]. Let $\mu = \mathbb{E}[\sum X_i]$ and $\sigma^2 = \text{Var}[\sum X_i]$.

$$\Pr\left(\left|\sum_{i=1}^{n} X_i - \mu\right| \geq s\sigma\right) \leq 2\exp\left(-\frac{s^2}{4}\right).$$

Can plot this bound for different s:



Looks a lot like a Gaussian (normal) distribution.

$$\mathcal{N}(0, \sigma^2) \text{ has density } p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{x^2}{2\sigma^2}}.$$

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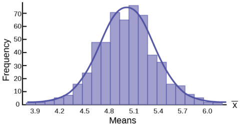
Exercise: Using this can show that for $\mathbf{X} \sim \mathcal{N}(0,\sigma^2)$: for any $s \geq 0$,

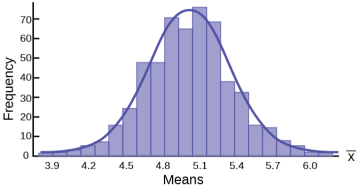
$$\Pr\left(|\mathbf{X}| \geq s \cdot \sigma\right) \leq O(1) \cdot e^{-\frac{s^2}{2}}.$$

Essentially the same bound that Bernstein's inequality gives!

Central Limit Theorem Interpretation: Bernstein's inequality gives a quantitative version of the CLT. The distribution of the sum of *bounded* independent random variables can be upper bounded with a Gaussian (normal) distribution.



Stronger Central Limit Theorem: The distribution of the sum of *n bounded* independent random variables converges to a Gaussian (normal) distribution as *n* goes to infinity.



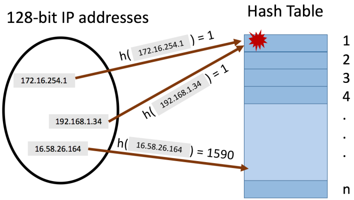
- Why is the Gaussian distribution is so important in statistics, science, ML, etc.?
- Many random variables can be approximated as the sum of a large number of small and roughly independent random effects. Thus, their distribution looks Gaussian by CLT.

A useful variation of the Bernstein inequality for binary (indicator) random variables is:

Chernoff Bound (simplified version): Consider independent random variables $X_1, \ldots, X_n$ taking values in $\{0, 1\}$. Let $\mu = \mathbb{E}[\sum_{i=1}^n X_i]$. For any $\delta \geq 0$

$$\Pr\left(\left|\sum_{i=1}^n X_i - \mu\right| \geq \delta\mu\right) \leq 2\exp\left(-\frac{\delta^2\mu}{2+\delta}\right).$$

As δ gets larger and larger, the bound falls of exponentially fast.



We hash m values $x_1, \ldots, x_m$ using a random hash function into a table with $n = m$ entries.

- I.e., for all $i \in [m]$ and $j \in [m]$, $\Pr(\mathbf{h}(x_i) = j) = \frac{1}{m}$ and hash values are chosen independently.

What will be the maximum number of items hashed into the same location? Give a bound that holds with probability $\geq 99/100$. Ok to ignore constant factors.

Chernoff Bound: Consider independent random variables $X_1, \ldots, X_n$ taking values in $\{0,1\}$. Let $\mu = \mathbb{E}[\sum_{i=1}^n X_i]$. For any $\delta \geq 0$

$$\Pr\left(\left|\sum_{i=1}^n X_i - \mu\right| \geq \delta\mu\right) \leq 2\exp\left(-\frac{\delta^2\mu}{2+\delta}\right).$$

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- I.e., for all $i \in [m]$ and $j \in [m]$, $\Pr(\mathbf{h}(x_i) = j) = \frac{1}{m}$ and hash values are chosen independently.

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What will be the maximum number of items hashed into the same location? Give a bound that holds with probability $\geq 99/100$.

Let $\mathbf{S}_i$ be the number of items hashed into position i and $\mathbf{S}_{i,j}$ be 1 if x_j is hashed into bucket i ($\mathbf{h}(x_j) = i$) and 0 otherwise.

$$\mathbb{E}[\mathbf{S}_i] = \sum_{j=1}^{m} \mathbb{E}[\mathbf{S}_{i,j}] = m \cdot \frac{1}{m} = 1.$$

By the Chernoff Bound: (with $\mu = 1$) for any $\delta \geq 0$,

$$\Pr(\mathbf{S}_i \geq 1 + \delta) \leq \Pr\left(\left|\sum_{i=1}^{n} \mathbf{S}_{i,j} - 1\right| \geq \delta\right) \leq 2\exp\left(-\frac{\delta^2}{2+\delta}\right)$$

m: total number of items hashed and size of hash table. $x_1, \ldots, x_m$: the items.
$\mathbf{h}$: random hash function mapping $x_1, \ldots, x_m \to [m]$.

$$\Pr(\mathsf{S}_i \geq 1+\delta) \leq \Pr\left(\left|\sum_{i=1}^{n} \mathsf{S}_{i,j} - 1\right| \geq \delta\right) \leq 2\exp\left(-\frac{\delta^2}{2+\delta}\right).$$

Set $\delta = 20\log m$. Gives:

$$\Pr(\mathsf{S}_i \geq 20\log m + 1) \leq 2\exp\left(-\frac{(20\log m)^2}{2+20\log m}\right) \leq \exp(-18\log m) \leq \frac{2}{m^{18}}.$$

Apply Union Bound:

$$\Pr(\max_{i\in[m]} \mathsf{S}_i \geq 20\log m + 1) = \Pr\left(\bigcup_{i=1}^{m} (\mathsf{S}_i \geq 20\log n + 1)\right)$$

$$\leq \sum_{i=1}^{m} \Pr(\mathsf{S}_i \geq 20\log m + 1) \qquad .$$

m: total number of items hashed and size of hash table. S_i: number of items hashed to bucket i. $\mathsf{S}_{i,j}$: indicator if x_j is hashed to bucket i. δ: any value ≥ 0.

$$\Pr(\mathbf{S}_i \geq 1+\delta) \leq \Pr\left(\left|\sum_{i=1}^{n} \mathbf{S}_{i,j} - 1\right| \geq \delta\right) \leq 2\exp\left(-\frac{\delta^2}{2+\delta}\right).$$

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Apply Union Bound:

$$\Pr(\max_{i\in[m]} \mathbf{S}_i \geq 20\log m + 1) = \Pr\left(\bigcup_{i=1}^{m} (\mathbf{S}_i \geq 20\log n + 1)\right)$$

$$\leq \sum_{i=1}^{m} \Pr(\mathbf{S}_i \geq 20\log m + 1) \leq m \cdot \frac{2}{m^{18}} = \frac{2}{m^{17}}.$$

m: total number of items hashed and size of hash table. $\mathbf{S}_i$: number of items hashed to bucket i. $\mathbf{S}_{i,j}$: indicator if x_j is hashed to bucket i. δ: any value ≥ 0.

Upshot: If we randomly hash m items into a hash table with m entries the maximum load per bucket is $O(\log m)$ with very high probability.

- So, even with a simple linked list to store the items in each bucket, worst case query time is $O(\log m)$.
- Using Chebyshev's inequality could only show the maximum load is bounded by $O(\sqrt{m})$ with good probability.
- The Chebyshev bound holds even with a pairwise independent hash function. The stronger Chernoff-based bound can be shown to hold with a k-wise independent hash function for $k = O(\log m)$.

Questions?

This concludes probability review/concentration bounds.