COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

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University of Massachusetts Amherst. Fall 2019.

Lecture 24 (Final Lecture!)

- Problem Set 4 due Sunday 12/15 at 8pm.
- Exam prep materials (including practice problems) posted under the 'Schedule' tab of the course page.
- I will hold office hours on both Tuesday and Wednesday next week from **10am to 12pm** to prep for final.
- SRTI survey is open until 12/22. Your feedback this semester has been very helpful to me, so please fill out the survey!
- `https://owl.umass.edu/partners/courseEvalSurvey/uma/`

Last Class:

- Compressed sensing and sparse recovery.
- Applications to sparse regression, frequent elements problem, sparse Fourier transform.

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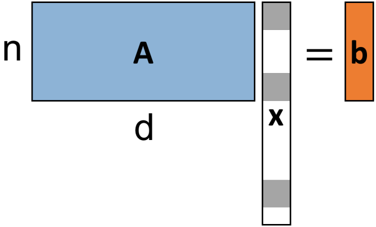
This Class:

- Finish up sparse recovery.
- Solution via basis pursuit. Idea of convex relaxation.
- Wrap up.

Problem Set Up: Given data matrix $\mathbf{A} \in \mathbb{R}^{n \times d}$ with $n < d$ and measurements $\mathbf{b} = \mathbf{A}\mathbf{x}$. Recover $\mathbf{x}$ under the assumption that it is k-sparse, i.e., has at most $k \ll d$ nonzero entries.



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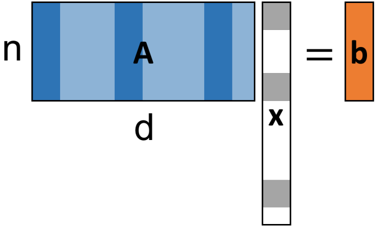

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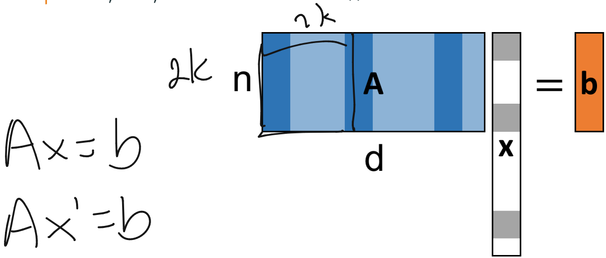
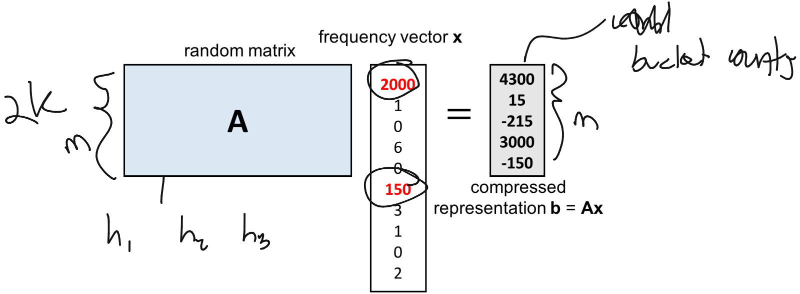


Last Time: Proved this is possible (i.e., the solution $\mathbf{x}$ is unique) when $\mathbf{A}$ has Kruskal rank $\geq 2k$.

$$\mathbf{x} = \arg\min_{\mathbf{z} \in \mathbb{R}^d : \mathbf{A}\mathbf{z} = \mathbf{b}} \|\mathbf{z}\|_0,$$

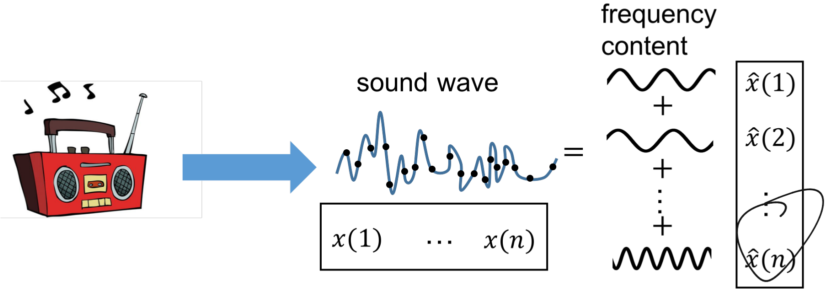
very hard.

Kruskal rank condition can be satisfied with n as small as $2k$



- A frequency vector with k out of n very frequent items is approximately k-sparse.
- Can be approximately recovered from its multiplication with a random matrix **A** with just $m = \tilde{O}(k)$ rows.
- $\mathbf{b} = \mathbf{A}\mathbf{x}$ can be maintained in a stream using just $O(m)$ space.
- Exactly the set up of Count-min sketch in linear algebraic notation.

Discrete Fourier Transform: For a discrete signal (aka a vector) $\mathbf{x} \in \mathbb{R}^n$, its discrete Fourier transform is denoted $\widehat{\mathbf{x}} \in \mathbb{C}^n$ and given by $\widehat{\mathbf{x}} = \mathbf{F}\mathbf{x}$, where $\mathbf{F} \in \mathbb{C}^{n \times n}$ is the discrete Fourier transform matrix.



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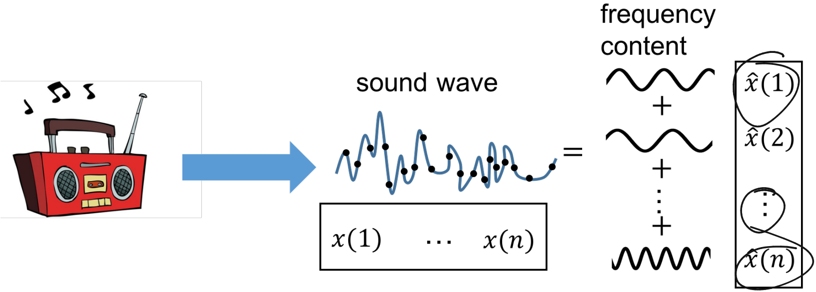

For many natural signals $\widehat{\mathbf{x}}$ is approximately sparse: a few dominant frequencies in a recording, superposition of a few radio transmitters sending at different frequencies, etc.

When the Fourier transform $\widehat{x}$ is sparse, can recover it from few measurements of x using sparse recovery.

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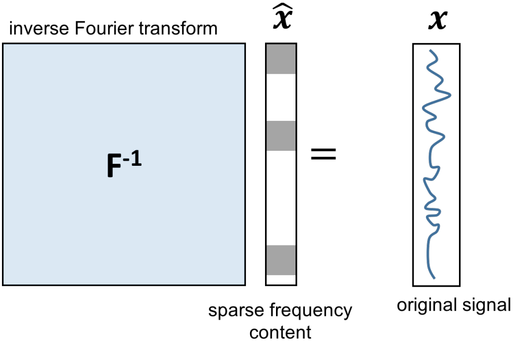

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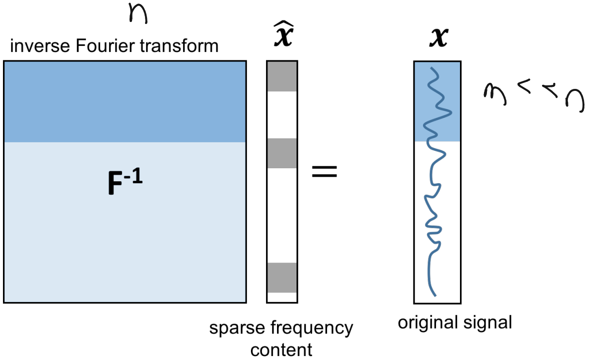

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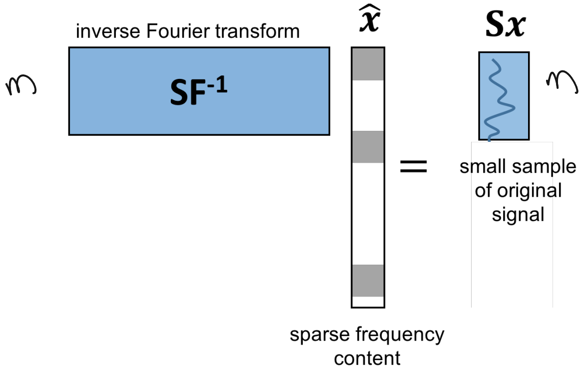

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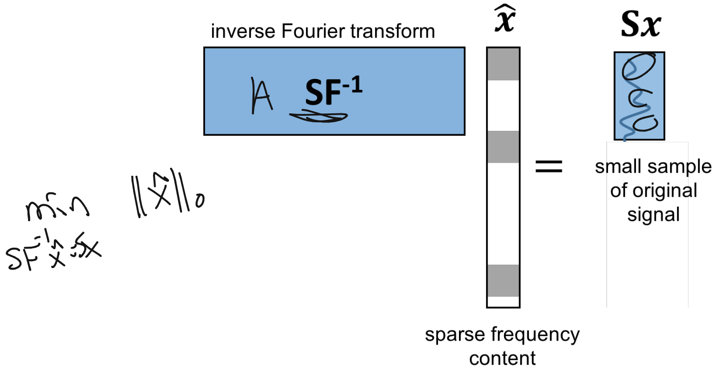

Translates to big savings in acquisition costs and number of sensors.

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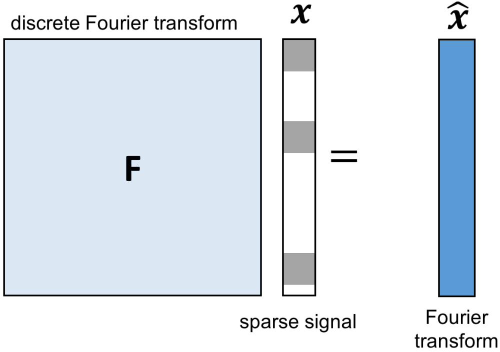

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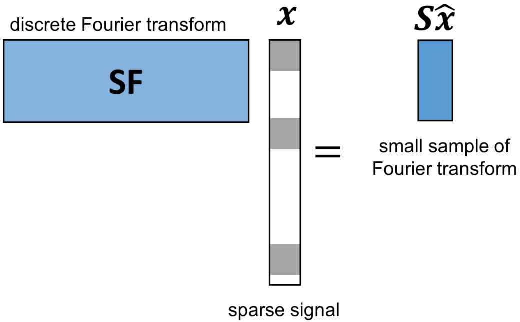

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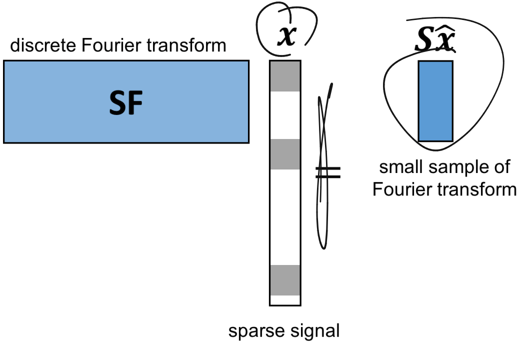

How do we access/measure entries of $S\hat{x}$?

- In seismology, $\mathbf{x}$ is an image of the earth's crust, and often sparse (e.g., a few locations of oil deposits).
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- Achieved by inducing vibrations of different frequencies with a vibroseis truck, air guns, explosions, etc and recording the response (more complicated in reality...)

Back to Algorithms

We would like to recover k-sparse $\mathbf{x}$ from measurements $\mathbf{b} = \mathbf{A}\mathbf{x}$ by solving the non-convex optimization problem:

$$\mathbf{x} = \operatorname*{arg\,min}_{\mathbf{z}\in\mathbb{R}^d:\mathbf{A}\mathbf{z}=\mathbf{b}} \|\mathbf{z}\|_0$$

Works if $\mathbf{A}$ has Kruskal rank $\geq 2k$, but very hard computationally.

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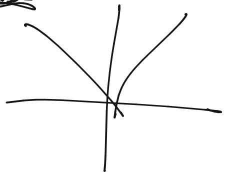

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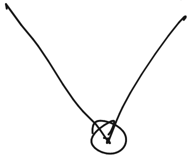

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- Projected (sub)gradient descent – convex objective function and convex constraint set.
- An instance of linear programming, so typically faster to solve with a linear programming algorithm (e.g., simplex, interior point).

Why should we hope that the basis pursuit solution returns the unique k-sparse $\mathbf{x}$ with $\mathbf{A}\mathbf{x} = \mathbf{b}$? The minimizer $\mathbf{z}^*$ will have small ℓ_1 norm but why would it even be sparse?

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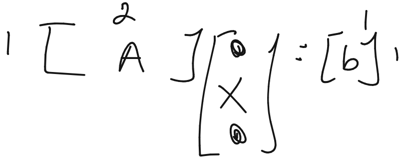
Assume that $n = 1, d = 2, k = 1$. So $\mathbf{A} \in \mathbb{R}^{1 \times 2}$ and $\mathbf{x} \in \mathbb{R}^2$ is 1-sparse.



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basis pursuit

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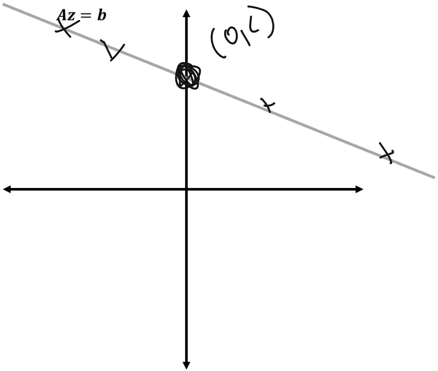

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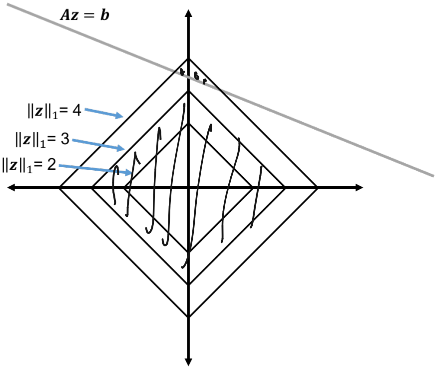

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3.6 = min ||z||1

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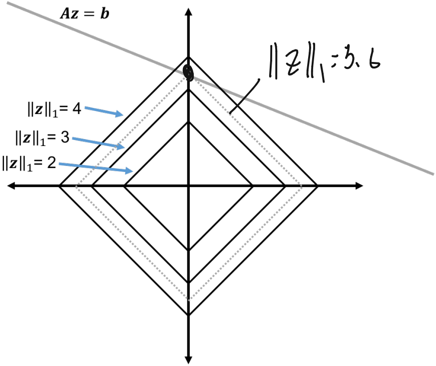

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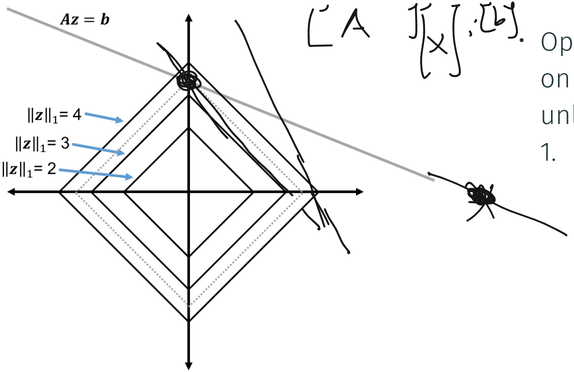

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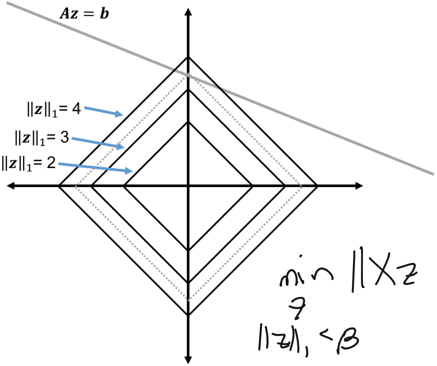

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- Similar intuition to the LASSO method.

$$\min_{z} \|Xz - y\|_2^2 + \|z\|_1$$

$\|z\|_1 < \beta$

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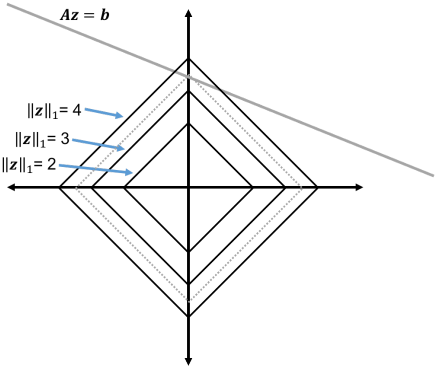

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Why should we hope that the basis pursuit solution returns the unique k-sparse $\mathbf{x}$ with $\mathbf{Ax} = \mathbf{b}$? The minimizer $\mathbf{z}^*$ will have small ℓ_1 norm but why would it even be sparse?

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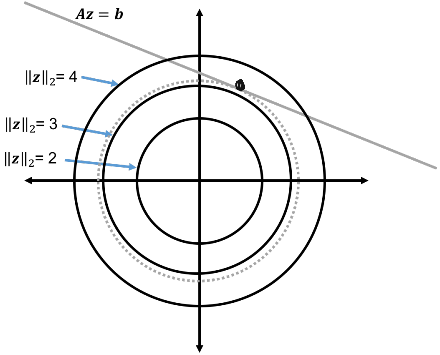

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Can prove that basis pursuit outputs the exact k-sparse solution $\mathbf{x}$ with $\mathbf{A}\mathbf{x} = \mathbf{b}$ (i.e, $\arg\min_{\mathbf{z}\in\mathbb{R}^d:\mathbf{A}\mathbf{z}=\mathbf{b}} \|\mathbf{z}\|_1 = \arg\min_{\mathbf{z}\in\mathbb{R}^d:\mathbf{A}\mathbf{z}=\mathbf{b}} \|\mathbf{z}\|_0$)

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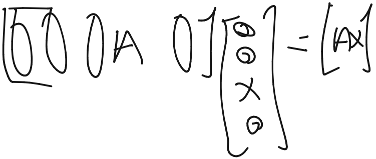
Kruskal rank ≥ m

> **Definition:** $\mathbf{A} \in \mathbb{R}^{n\times d}$ has the (m, ϵ) restricted isometry property (is (m, ϵ)-RIP) if for all m-sparse vectors $\mathbf{x}$:
>
> $$(1-\epsilon)\|\mathbf{x}\|_2 \leq \|\mathbf{A}\mathbf{x}\|_2 \leq (1+\epsilon)\|\mathbf{x}\|_2$$



$\|Ax\|_2 > 0$

Can prove that basis pursuit outputs the exact k-sparse solution $\mathbf{x}$ with $\mathbf{A}\mathbf{x} = \mathbf{b}$ (i.e, $\arg\min_{\mathbf{z}\in\mathbb{R}^d:\mathbf{A}\mathbf{z}=\mathbf{b}} \|\mathbf{z}\|_1 = \arg\min_{\mathbf{z}\in\mathbb{R}^d:\mathbf{A}\mathbf{z}=\mathbf{b}} \|\mathbf{z}\|_0$)

- Requires a strengthening of the Kruskal rank $\geq 2k$ assumption (that still holds in many applications).

Definition: $\mathbf{A} \in \mathbb{R}^{n\times d}$ has the (m, ϵ) restricted isometry property (is (m, ϵ)-RIP) if for all m-sparse vectors $\mathbf{x}$:

$$(1-\epsilon)\|\mathbf{x}\|_2 \leq \|\mathbf{A}\mathbf{x}\|_2 \leq (1+\epsilon)\|\mathbf{x}\|_2$$

Theorem: If $\mathbf{A}$ is $(3k, \epsilon)$-RIP for small enough constant ϵ, then $\mathbf{z}^\star = \arg\min_{\mathbf{z}\in\mathbb{R}^d:\mathbf{A}\mathbf{z}=\mathbf{b}} \|\mathbf{z}\|_1$ is equal to the unique k-sparse $\mathbf{x}$ with $\mathbf{A}\mathbf{x} = \mathbf{b}$ (i.e., basis pursuit solves the sparse recovery problem).

Wrap Up

Thanks for a great semester!

Randomization as a computational resource for massive datasets.

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- Focus on problems that are easy on small datasets but hard at massive scale – set size estimation, load balancing, distinct elements counting (MinHash), checking set membership (Bloomfilters), frequent items counting (Count-min sketch), near neighbor search (locality sensitive hashing).

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- Just the tip of the iceberg on randomized streaming/sketching/hashing algorithms.
- In the process covered probability/statistics tools that are very useful beyond algorithm design: concentration inequalities, higher moment bounds, law of large numbers, central limit theorem, linearity of expectation and variance, union bound, median as a robust estimator.

Methods for working with (compressing) high-dimensional data

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- Low-rank structure in graphs – nonlinear dimensionality reduction and spectral clustering for community detection, stochastic block model, matrix concentration.
- In the process covered linear algebraic tools that are very broadly useful in ML and data science: eigendecomposition, singular value decomposition, projection, norm transformations.

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- Online optimization, online gradient descent, and how to use it to analyze stochastic gradient descent (by far the most common optimization method in ML).
- Lots that we didn't cover: accelerated methods, adaptive methods, second order methods (quasi-Newton methods), practical considerations. Hopefully gave mathematical tools to understand these methods.

GRAB-BAG TOPICS

- The weirdness of high-dimensional space and geometry. Connections to randomized methods, dimensionality reduction. Always useful to keep in mind.
- Compressed sensing/sparse recovery – a very broad and widely-used framework for working with high-dimensional data. Connection to streaming algorithms (frequent items counting) and convex optimization.

Thanks!