COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

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University of Massachusetts Amherst. Fall 2019.

Lecture 23

- Problem Set 4 due Sunday 12/15 at 8pm.
- Exam prep materials posted under the ‘Schedule’ tab of the course page.
- SRTI survey is open until 12/22. Your feedback this semester has been very helpful to me, so please fill out the survey!
- `https://owl.umass.edu/partners/courseEvalSurvey/uma/`

Last Class:

- Some counterintuitive properties of high dimensional space.
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This Class:

- Compressed sensing and sparse recovery.
- Applications to sparse regression, frequent elements problem, sparse Fourier transform, efficient imaging, etc.

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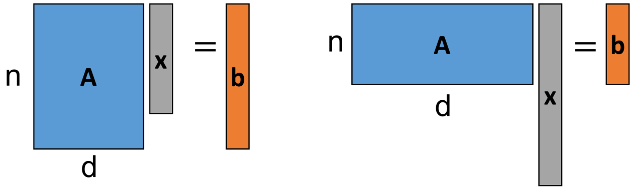

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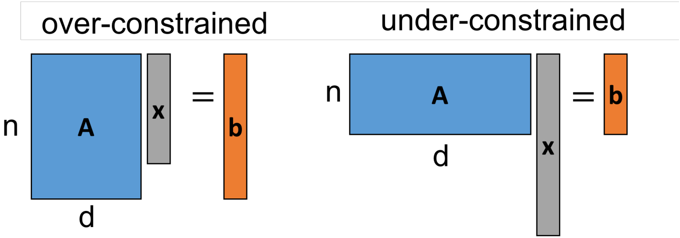

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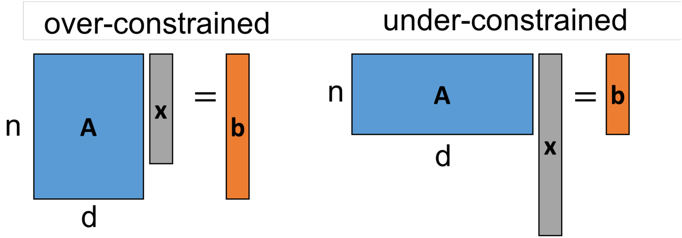
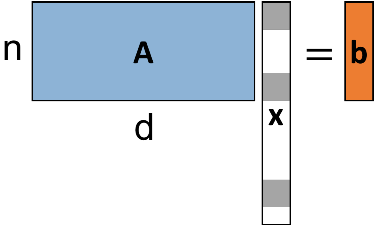


Compressed sensing: Under what assumptions we can still find $\mathbf{x}$ when the number of 'measurements' n is smaller than the number of features d (i.e., when $\mathbf{b}$ is a compression of $\mathbf{x}$)?

The most common assumption of compressed sensing is that $\mathbf{x}$ is *k-sparse*, i.e., has at most $k \ll d$ nonzero entries.



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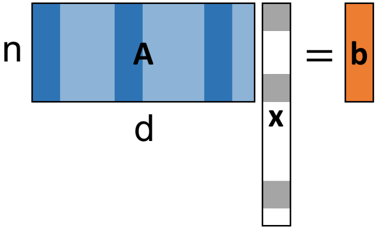

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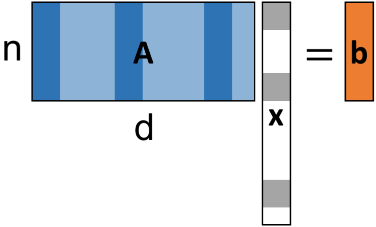

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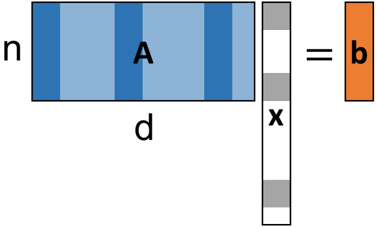

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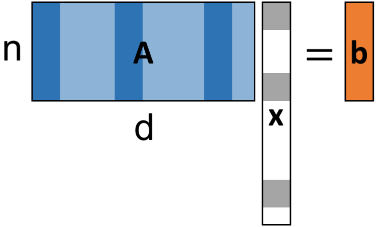

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First: Under what condition can you find **x** assuming knowledge that it is at most k-sparse? When every set of $2k$ columns in **A** is linearly independent – i.e., **A** has Kruskal rank $2k$.

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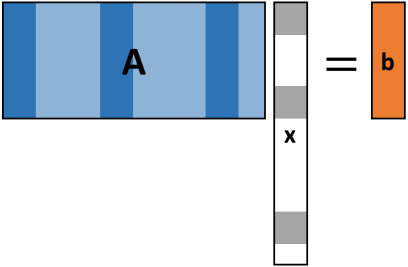

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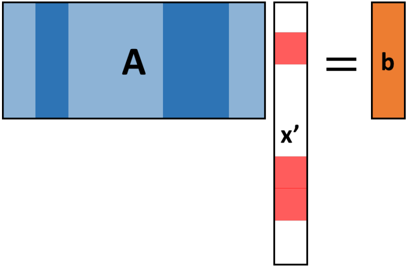

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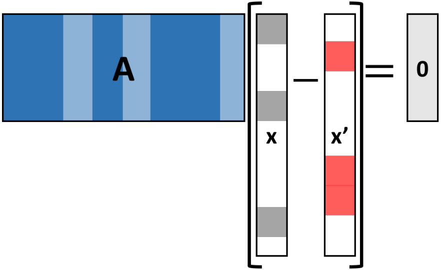

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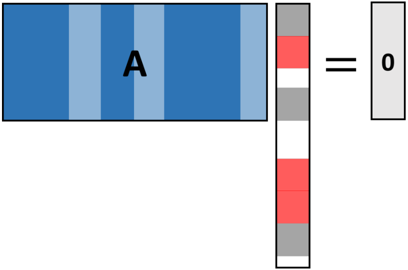

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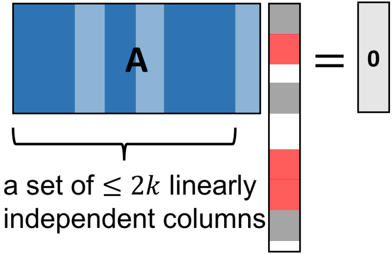

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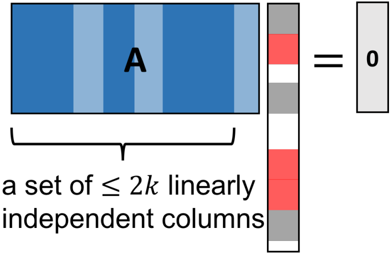

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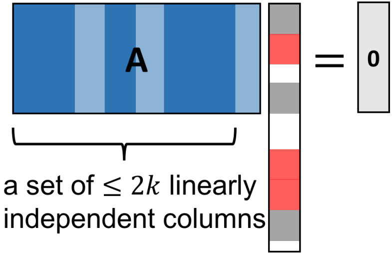

- Thus $\mathbf{x}$ is the unique k-sparse solution to $\mathbf{A}\mathbf{x} = \mathbf{b}$.

To satisfy the Kruskal rank $\geq 2k$ assumption $\mathbf{A}$ just needs $2k$ rows (compared with d rows to have full column rank). Can recover a d-dimensional k-sparse vector $\mathbf{x}$ from just $2k$ measurements $\mathbf{b} = \mathbf{A}\mathbf{x}$.

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A major accomplishment of compressed sensing/sparse recovery is to make the above procedure efficient and noise robust.

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- The techniques discussed carry over to the noisy setting.
- Generally won't find $\mathbf{x}$ exactly, but up to some good approximation.

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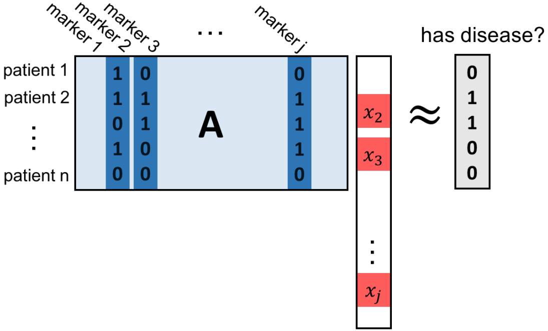

SPARSE REGRESSION

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- Want to find a linear regression model $Ax \approx b$ that only uses a small number of features (x is sparse).
- Interesting even in the over-constrained case. Often talked about as a different problem than compressed sensing, but very related.

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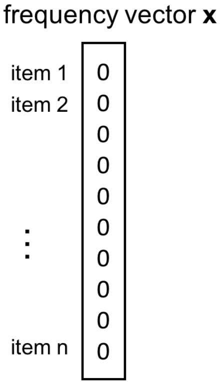
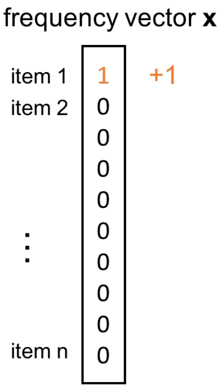
- In this setting, the problem is solved with sparse recovery techniques.

FREQUENT ITEMS COUNTING




frequency vector x
item 1
item 2
⋮
item n
1
0
0
0
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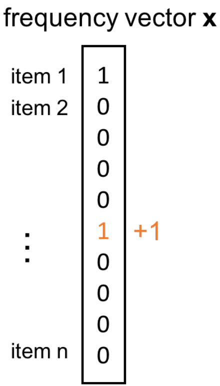
0
0
+1


frequency vector **x**
item 1
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item n
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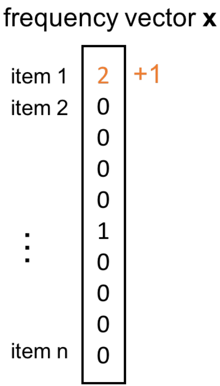
frequency vector **x**
item 1
item 2
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item n
2
+1
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0
0
1
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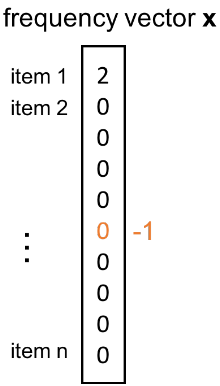
0


frequency vector x
item 1
item 2
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item n
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-1
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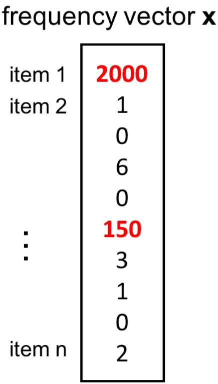
0


frequency vector **x**
item 1
item 2
⋮
item n
2000
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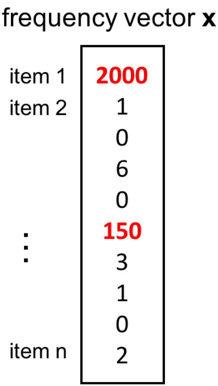
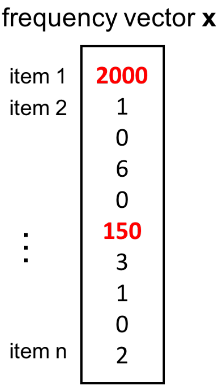
6
0
150
3
1
0
2



- Storing $\mathbf{x}$ requires $O(n)$ space. Will instead store $\mathbf{A}\mathbf{x}$ where $\mathbf{A} \in \mathbb{R}^{O(k)\times n}$ is a random matrix.



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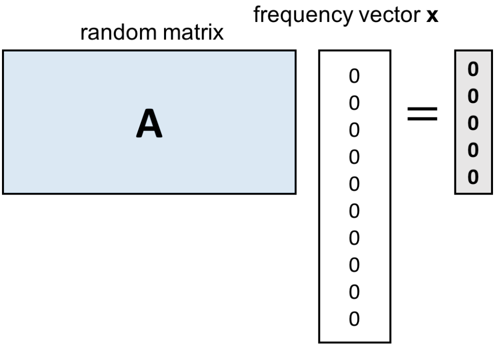

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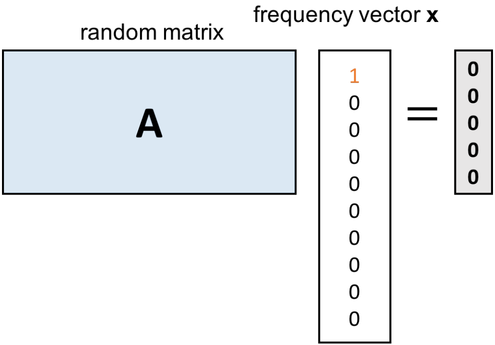

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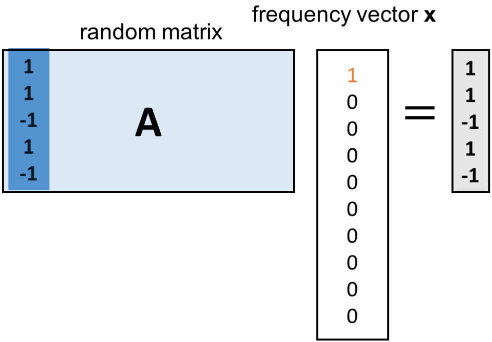

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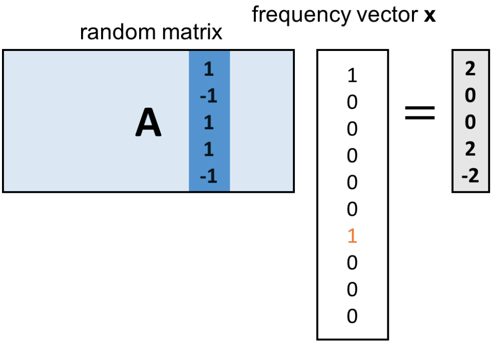
FREQUENT ITEMS COUNTING



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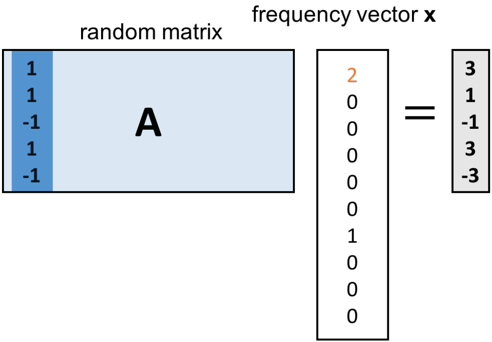

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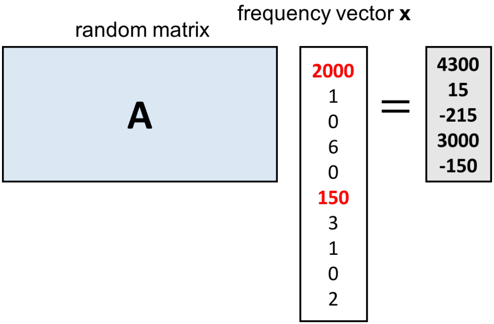

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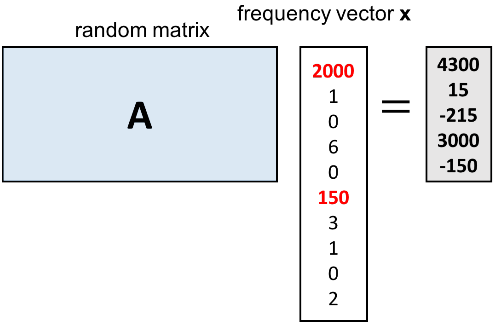

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- If there are a k heavy items, **x** is approximately k-sparse.
- Estimating the large entries of **x** (the counts of the most frequent items) from the compression **Ax** is exactly sparse recovery.

Many of the most important applications of sparse recovery are in imaging and signal processing.

- Many signals are sparse in some basis (Fourier, wavelet, etc.).
- Using sparse recovery techniques, an n pixel image/n point signal can thus be recovered from many fewer than n measurements.
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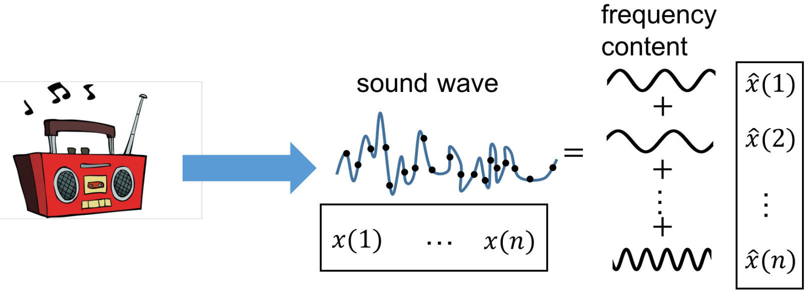
In general, there are a lot of practical complexities here. So everything I say is a major oversimplification.

Discrete Fourier Transform: For a discrete signal (aka a vector) $\mathbf{x} \in \mathbb{R}^n$, its discrete Fourier transform is denoted $\widehat{\mathbf{x}} \in \mathbb{C}^n$ and given by $\widehat{\mathbf{x}} = \mathbf{F}\mathbf{x}$, where $\mathbf{F} \in \mathbb{C}^{n \times n}$ is the discrete Fourier transform matrix.



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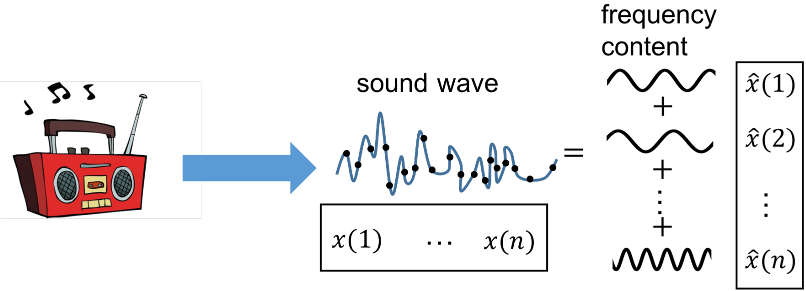

For many natural signals $\widehat{\mathbf{x}}$ is approximately sparse: a few dominant frequencies in a recording, superposition of a few radio transmitters sending at different frequencies, etc.

When the Fourier transform $\widehat{x}$ is sparse, can recover x from few measurements using sparse recovery.

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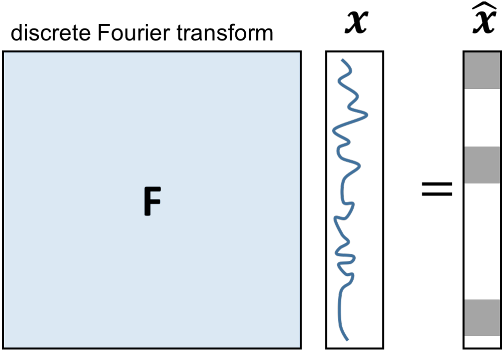

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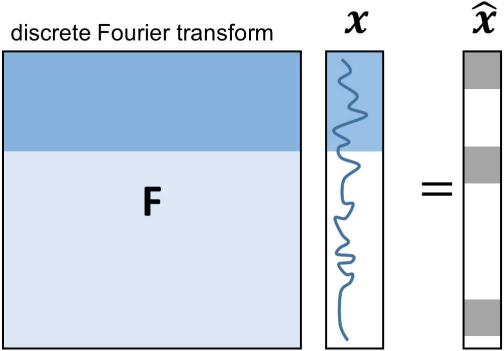

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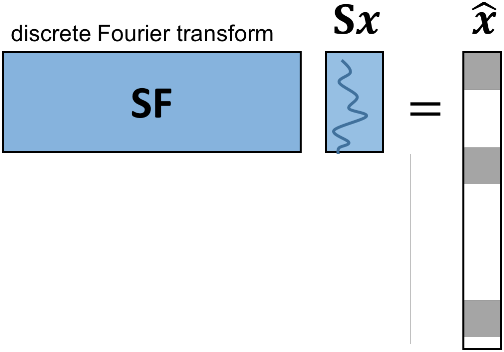

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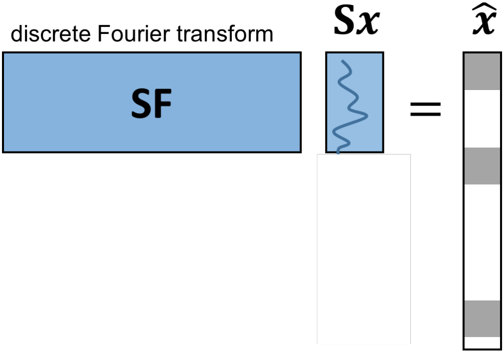

Translates to big savings in acquisition costs, the number of sensors required, etc.

Back to Algorithms

We would like to recover k-sparse $\mathbf{x}$ from measurements $\mathbf{b} = \mathbf{A}\mathbf{x}$ by solving the non-convex optimization problem:

$$\mathbf{x} = \operatorname*{arg\,min}_{\mathbf{z} \in \mathbb{R}^d : \mathbf{A}\mathbf{z} = \mathbf{b}} \|\mathbf{z}\|_0$$

Works if $\mathbf{A}$ has Kruskal rank $\geq 2k$, but very hard computationally.

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- An instance of linear programming, so typically faster to solve with a linear programming algorithm.

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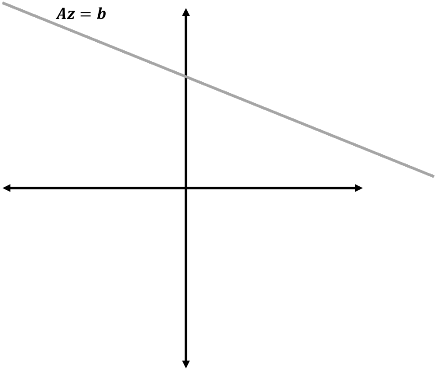

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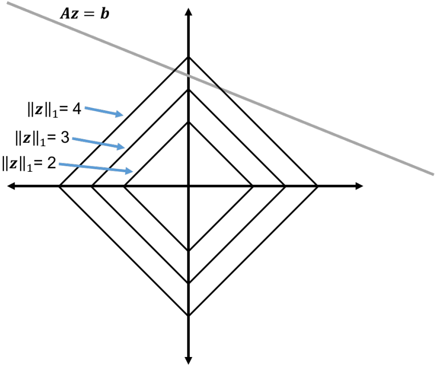

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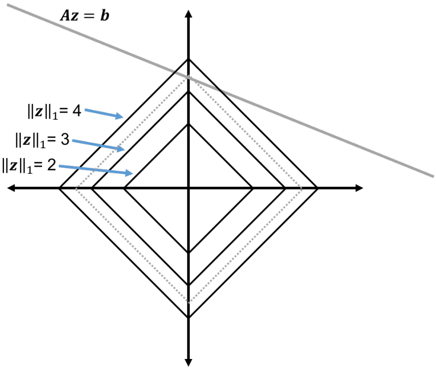

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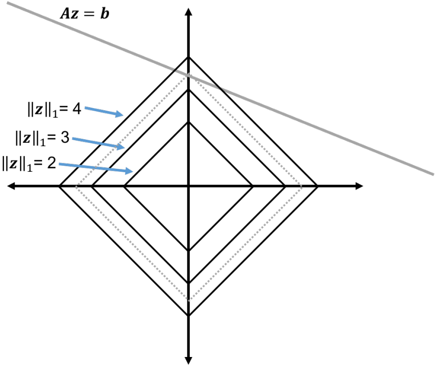

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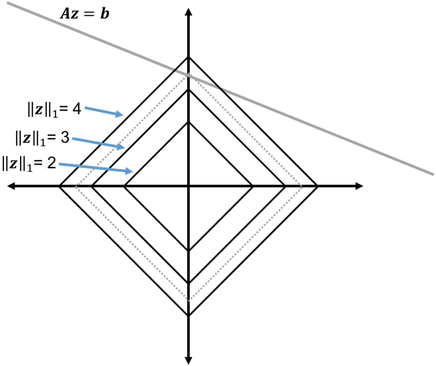

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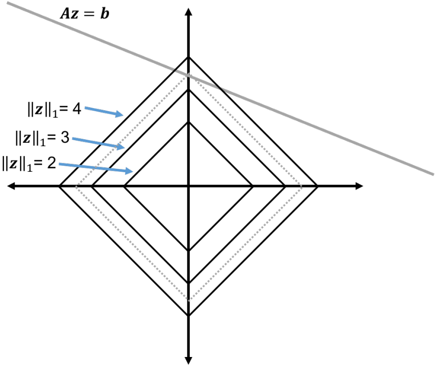

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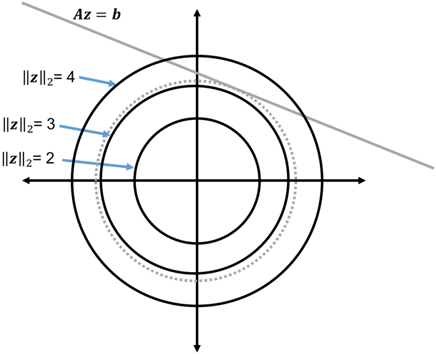

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BASIS PURSUIT THEOREM

Can prove that basis pursuit outputs the exact k-sparse solution $\mathbf{x}$ with $\mathbf{A}\mathbf{x} = \mathbf{b}$ (same as $\arg\min_{\mathbf{z}\in\mathbb{R}^d:\mathbf{A}\mathbf{z}=\mathbf{b}} \|\mathbf{z}\|_0$)

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Theorem: If $\mathbf{A}$ is $(3k, \epsilon)$-RIP for small enough constant ϵ, then $\mathbf{z}^\star = \arg\min_{\mathbf{z}\in\mathbb{R}^d:\mathbf{A}\mathbf{z}=\mathbf{b}} \|\mathbf{z}\|_1$ is equal to the unique k-sparse $\mathbf{x}$ with $\mathbf{A}\mathbf{x} = \mathbf{b}$ (i.e., basis pursuit solves the sparse recovery problem).

Questions?