COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

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University of Massachusetts Amherst. Fall 2019.

Lecture 22

LOGISTICS

- Problem Set 4 released last night. Due Sunday 12/15 at 8pm.
- Final Exam Thursday 12/19 at 10:30am in Thompson 104.
- Exam prep materials (list of topics, practice problems) coming in next couple of days.

Before Break:

- Finished discussion of SGD.
- Gradient descent and SGD as applied to least squares regression.

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This Class:

- A quick tour of the counterintuitive properties of high-dimensional space.
- Many connections to concentration inequalities.
- Implications for working with high-dimensional data (curse of dimensionality).

Modern data analysis often involves very high-dimensional data points.

- Websites record (tens of) thousands of measurements per user: who they follow, when they visit the site, timestamps for specific iteractions, etc.
- A 3 minute, 500×500 pixel video clip at 15 FPS has ≥ 2 billion pixel values.
- The human genome has 3 billion+ base pairs.

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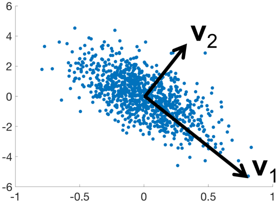
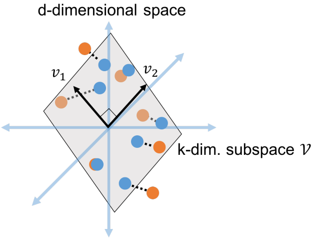
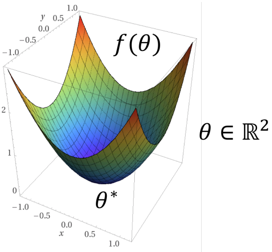
Typically when discussing algorithm design we imagine data in much lower (usually 3) dimensional space.



6
4
2
0
-2
-4
-6
-1
-0.5
0
0.5
1
$\mathbf{v}_2$
$\mathbf{v}_1$

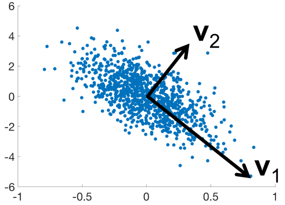


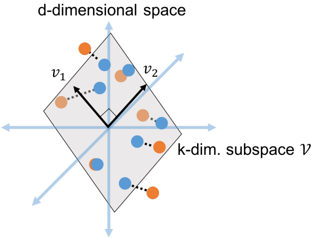
d-dimensional space
v_1
v_2
k-dim. subspace $\mathcal{V}$

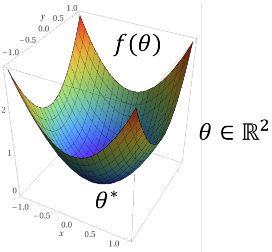


$f(\theta)$
θ^*
$\theta \in \mathbb{R}^2$

LOW-DIMENSIONAL INTUITION







This can be a bit dangerous as in reality high-dimensional space is very different from low-dimensional space.

What is the largest set of mutually orthogonal unit vectors in d-dimensional space?

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NEARLY ORTHOGONAL VECTORS

What is the largest set of unit vectors in d-dimensional space that have all pairwise dot products $|\langle \vec{x}, \vec{y} \rangle| \le \epsilon$? (think $\epsilon = .01$)

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1. d
2. $\Theta(d)$
3. $\Theta(d^2)$
4. $2^{\Theta(d)}$

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- x_i is always a unit vector.
- $\mathbb{E}[\langle x_i, x_j \rangle] = 0$.
- By a Chernoff bound, $\Pr[|\langle x_i, x_j \rangle| \geq \epsilon] \leq 2e^{-\epsilon^2 d/3}$.

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- If we chose $t = \frac{1}{2}e^{\epsilon^2 d/6}$, using a union bound over all $\leq t^2 = \frac{1}{4}e^{\epsilon^2 d/3}$ possible pairs, with probability $\geq 1/2$ all with be nearly orthogonal.

Up Shot: In d-dimensional space, a set of $2^{\Theta(\epsilon^2 d)}$ random unit vectors have all pairwise dot products at most ϵ (think $\epsilon = .01$)

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Curse of dimensionality for sampling/learning functions in high dimensional space – samples are very 'sparse' unless we have a huge amount of data.

- Only hope is if we have strong low-dimensional structure.

BIZARRE SHAPE OF HIGH-DIMENSIONAL BALLS

Let $\mathcal{B}_d$ be the unit ball in d dimensions. $\mathcal{B}_d = \{x \in \mathbb{R}^d : \|x\|_2 \leq 1\}$.

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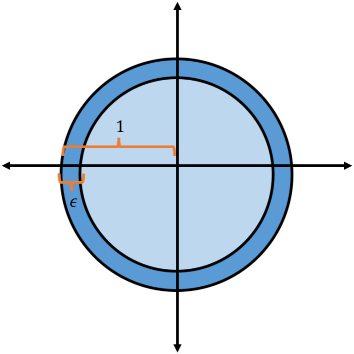
What percentage of the volume of $\mathcal{B}_d$ falls within ϵ distance of its surface?



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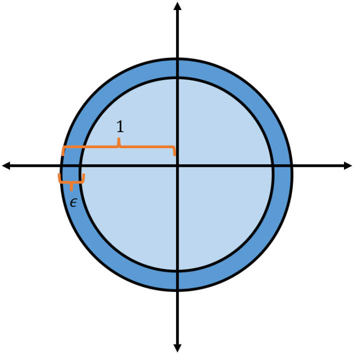

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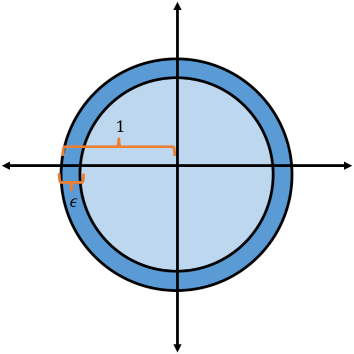
What percentage of the volume of $\mathcal{B}_d$ falls within ϵ distance of its surface? Answer: all but a $(1-\epsilon)^d \leq e^{-\epsilon d}$ fraction. Exponentially small in the dimension d!



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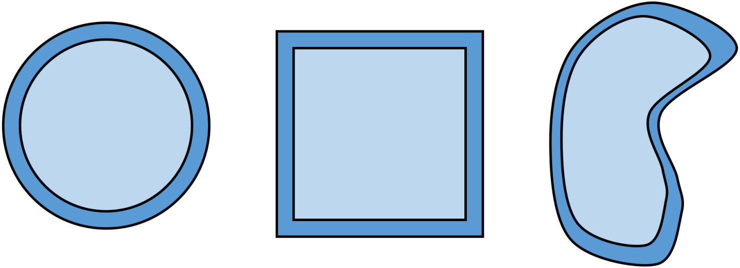
- **Isoperimetric inequality**: the ball has the maximum surface area/volume ratio of any shape.



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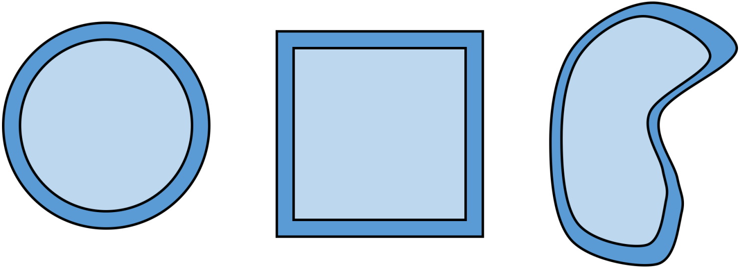

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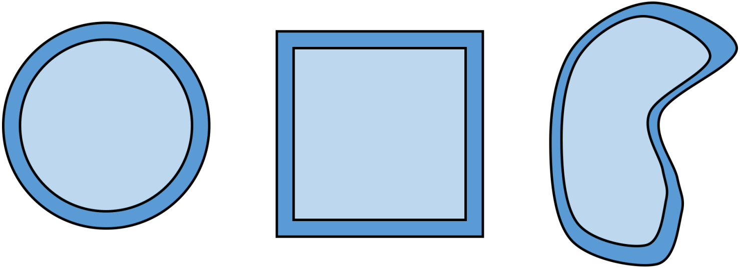
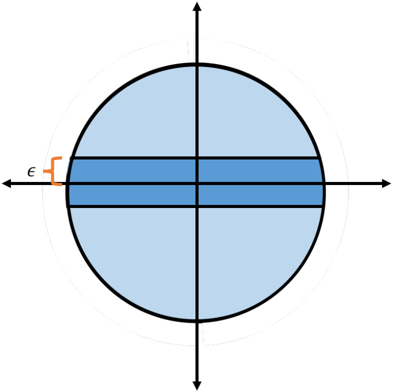


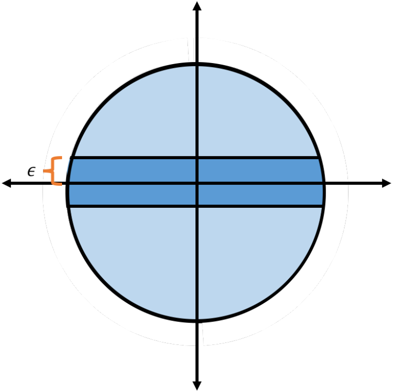
- If we randomly sample points from any high-dimensional shape, nearly all will fall near its surface.
- 'All points are outliers.'

What percentage of the volume of $\mathcal{B}_d$ falls within ϵ distance of its equator?



Formally: volume of set $S = \{x \in \mathcal{B}_d : |x(1)| \leq \epsilon\}$.

What percentage of the volume of $\mathcal{B}_d$ falls within ϵ distance of its equator? Answer: all but a $2^{\Theta(-\epsilon^2 d)}$ fraction.



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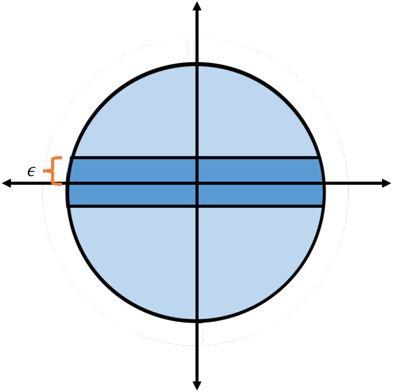

Formally: volume of set $S = \{x \in \mathcal{B}_d : |x(1)| \leq \epsilon\}$.

By symmetry, all but a $2^{\Theta(-\epsilon^2 d)}$ fraction of the volume falls within ϵ of any equator! $S = \{x \in \mathcal{B}_d : |\langle x, t\rangle| \leq \epsilon\}$

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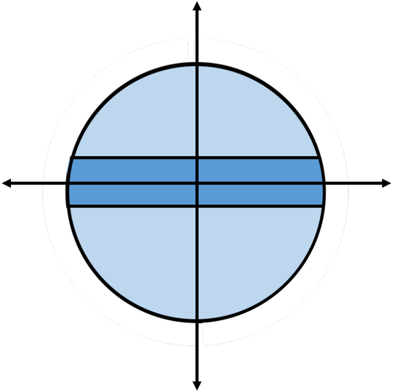

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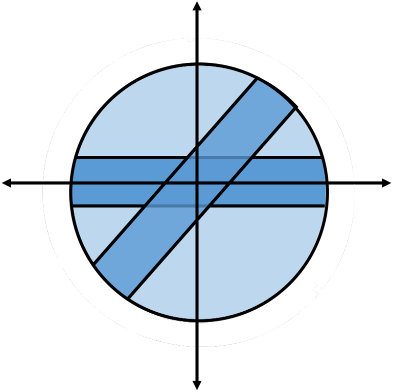

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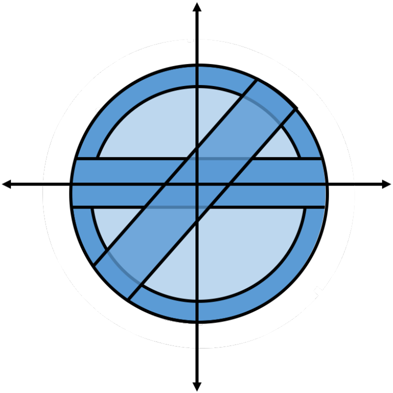

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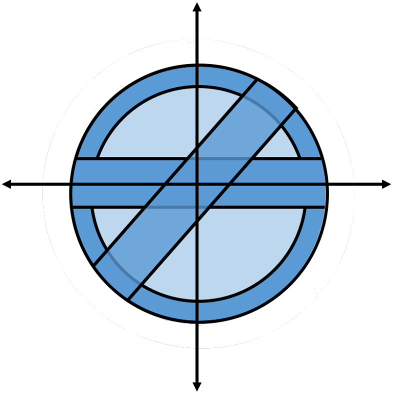

How is this possible?

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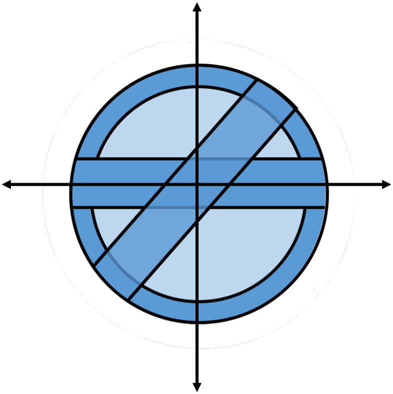

How is this possible? High-dimensional space looks nothing like this picture!

Claim: All but a $2^{\Theta(-\epsilon^2 d)}$ fraction of the volume of a ball falls within ϵ of its equator. I.e., in $S = \{x \in \mathcal{B}_d : |x(1)| \leq \epsilon\}$.

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- Let x have entries set to independent Gaussians $\mathcal{N}(0,1)$ and let $\bar{x} = \frac{x}{\|x\|_2}$. $\bar{x}$ is selected uniformly at random from the surface of the ball.

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CONCENTRATION OF VOLUME AT EQUATOR

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- Conditioning on $\|x\|_2^2 \geq d/2$, since $x(1)$ is normally distributed,

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- Conditioning on $\|x\|_2^2 \geq d/2$, since $x(1)$ is normally distributed,

$$\begin{aligned} \Pr[|\bar{x}(1)| > \epsilon] &= \Pr[|x(1)| > \epsilon \cdot \|x\|_2] \\ &\leq Pr[|x(1)| > \epsilon \cdot \sqrt{d/2}] = 2^{\Theta(-(\epsilon\sqrt{d/2})^2)} = 2^{\Theta(-\epsilon^2 d)}. \end{aligned}$$

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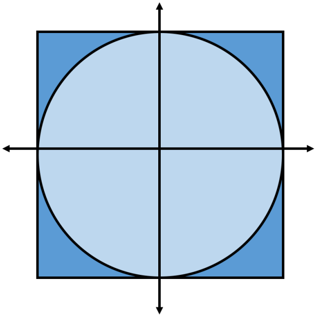

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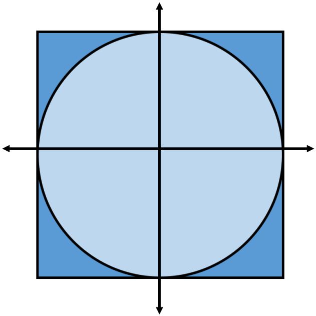

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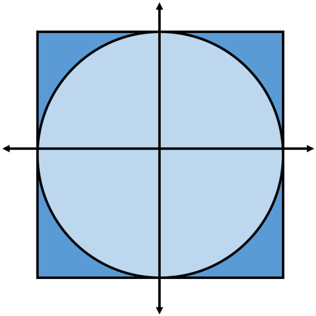

But volume of $\mathcal{C}_d$ is 2^d while volume of $\mathcal{B}^d$ is $\frac{\pi^{\frac{d}{2}}}{(d/2)!} = \frac{1}{d^{\Theta(d)}}$. A huge gap! So something is very different about these shapes...

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HIGH DIMENSIONAL CUBES

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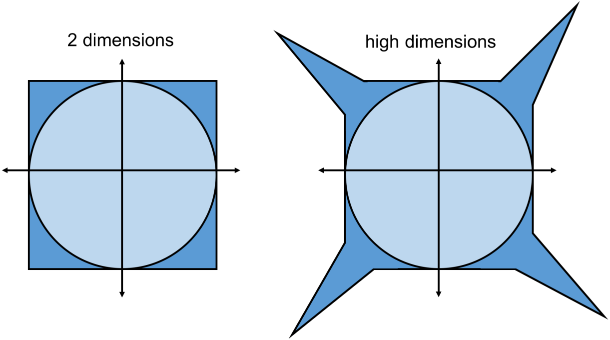

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Recall: The Johnson Lindenstrauss lemma states that if $\mathbf{\Pi} \in \mathbb{R}^{m \times d}$ is a random matrix (linear map) with $m = O\left(\frac{\log n}{\epsilon^2}\right)$, for $x_1, \ldots, x_n \in \mathbb{R}^d$ with high probability, for all i, j:

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But these different dimensional spaces have very different geometries, so how is this possible?

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- m is chosen just large enough so that the odd geometry of d-dimensional space will still hold on the n points in question.

Questions?