COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

Cameron Musco

University of Massachusetts Amherst. Fall 2019.

Lecture 20

LOGISTICS

- Problem Set 3 was due Friday/Sunday.
- Problem Set 4 will be on optimization. Out before Thanksgiving and due sometime towards the end of classes.
- Final is on December 19th, 10:30am-12:30pm.

Last Class:

- Analysis of gradient descent for optimizing convex functions.
- (The same) analysis of projected gradient descent for optimizing under constraints.

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This Class:

- Stochastic and online gradient descent for computationally efficient and online learning.
- Unified analysis.

Typical Optimization Problem in Machine Learning: Given data points $\vec{x}_1, \ldots, \vec{x}_n$ and labels/observations $y_1, \ldots, y_n$ solve:

$$\vec{\theta}^* = \arg\min_{\vec{\theta} \in \mathbb{R}^d} L(\vec{\theta}, \mathbf{X}) = \sum_{j=1}^{n} \ell(M_{\vec{\theta}}(\vec{x}_j), y_j).$$

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argmin

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$l_i(\vec{\theta})$

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GRADIENT DESCENT AT SCALE

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- If the data point is chosen uniformly at random, the sampled gradient is correct in expectation.

$x_1 \; . \; . \; . \; x_n \qquad x_j$

$$\vec{\nabla} L(\vec{\theta}, \mathbf{X}) = \sum_{i=j}^{n} \vec{\nabla} \ell(M_{\vec{\theta}}(\vec{x}_j), y_j) \rightarrow \mathbb{E}_{j \sim [n]}[\vec{\nabla} \ell(M_{\vec{\theta}}(\vec{x}_j), y_j)] = \frac{1}{n} \cdot \vec{\nabla} L(\vec{\theta}, \mathbf{X}).$$

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$$= \sum_{j=1}^{n} \frac{1}{n} \cdot \nabla \ell_j$$

- The key idea behind **stochastic gradient descent** (SGD).

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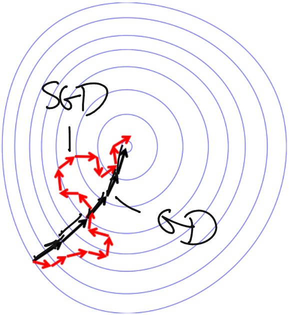

$$\underline{\vec{\theta}^{(i+1)} = \vec{\theta}^{(i)} - \eta \cdot \vec{\nabla} L(\vec{\theta}^{(i)}, \mathrm{X})} \text{ vs. } \vec{\theta}^{(i+1)} = \underline{\vec{\theta}^{(i)} - \eta \cdot \vec{\nabla} \ell(M_{\vec{\theta}^{(i)}}(\vec{x}_j)}, y_j)$$

ONLINE GRADIENT DESCENT

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- Spam filters are incrementally updated and adapt as they see more examples of spam over time.
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Will view SGD as a special case: when data points are presented (by design) in a random order.

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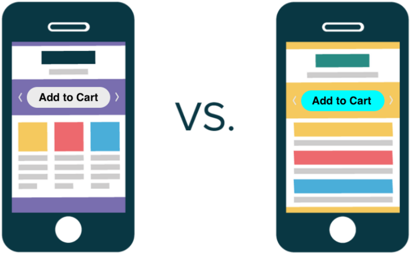
- At each step, first pick (play) a parameter vector $\vec{\theta}^{(i)}$.
- Then are told f_i and incur cost $f_i(\vec{\theta}^{(i)})$.
- **Goal:** Minimize total cost $\sum_{i=1}^{t} f_i(\vec{\theta}^{(i)})$.

$\theta^1 \ldots \theta^t$

No assumptions on how $f_1, \ldots, f_t$ are related to each other!

UI design via online optimization.



- Parameter vector $\vec{\theta}^{(i)}$: some encoding of the layout at step i.
- Functions $f_1, \ldots, f_t$: $f_i(\vec{\theta}^{(i)}) = 1$ if user does not click 'add to cart' and $f_i(\vec{\theta}^{(i)}) = 0$ if they do click.
- Want to maximize number of purchases. I.e., minimize $\sum_{i=1}^{t} f_i(\vec{\theta}^{(i)})$

Home pricing tools.

$\vec{x} = [\#baths, \#beds, \#floors \ldots]$

linear model

$\langle \vec{x}, \vec{\theta} \rangle$

$275,000

- Parameter vector $\vec{\theta}^{(i)}$: coefficients of linear model at step i.
- Functions $f_1, \ldots, f_t$: $f_i(\vec{\theta}^{(i)}) = (\cancel{\vec{\theta}}^{(i)} - price_i)^2$ revealed when $home_i$ is listed or sold. $(\langle \vec{x}_i, \theta^i \rangle - price_i)^2$
- Want to minimize total squared error $\sum_{i=1}^{t} f_i(\vec{\theta}^{(i)})$ (same as classic least squares regression).

In normal optimization, we seek $\hat{\theta}$ satisfying:

$$f(\hat{\theta}) \leq \min_{\vec{\theta}} f(\vec{\theta}) + \epsilon.$$

REGRET

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θ^*

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- This error metric is a bit 'unfair'. Why?
- Comparing online solution to best fixed solution in hindsight. ϵ can be negative!

Assume that: $(\langle x_i, \theta^i \rangle - y_i)^2$

- $f_1, \ldots, f_t$ are all convex.
- Each f_i is G-Lipschitz (i.e., $\|\vec{\nabla} f_i(\vec{\theta})\|_2 \leq G$ for all $\vec{\theta}$.)
- $\|\vec{\theta}^{(1)} - \vec{\theta}^{ol}\|_2 \leq R$ where $\theta^{(1)}$ is the first vector chosen.

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θ^{Ol} : $\arg\min_{\theta} \sum f_i(\theta)$

Online Gradient Descent

- Set step size $\eta = \frac{R}{G\sqrt{t}}$.
- For $i = 1, \ldots, t$
  - Play $\vec{\theta}^{(i)}$ and incur cost $f_i(\vec{\theta}^{(i)})$.
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Theorem – OGD on Convex Lipschitz Functions: For convex G-Lipschitz $f_1, \ldots, f_t$, OGD initialized with starting point $\theta^{(1)}$ within radius R of θ^{ol}, using step size $\eta = \frac{R}{G\sqrt{t}}$, has regret bounded by:

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Step 1.1: For all i, $\nabla f_i(\theta^{(i)})^T(\theta^{(i)} - \theta^{ol}) \leq \frac{\|\theta^{(i)} - \theta^{ol}\|_2^2 - \|\theta^{(i+1)} - \theta^{ol}\|_2^2}{2\eta} + \frac{\eta G^2}{2}$.

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ONLINE GRADIENT DESCENT ANALYSIS

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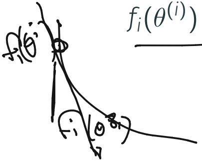
Convexity $\implies$ **Step 1:** For all i,

$$\underline{f_i(\theta^{(i)}) - f_i(\theta^{ol})} \leq \frac{\|\theta^{(i)} - \theta^{ol}\|_2^2 - \|\theta^{(i+1)} - \theta^{ol}\|_2^2}{2\eta} + \frac{\eta G^2}{2}.$$



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$\|\theta^1 - \theta^{ol}\|_2^2 - \|\theta^{t+1} \theta^{ol}\|_2^2$

$$\left[\sum_{i=1}^{t} f_i(\theta^{(i)}) - \sum_{i=1}^{t} f_i(\theta^{ol})\right] \leq \sum_{i=1}^{t} \frac{\|\theta^{(i)} - \theta^{ol}\|_2^2 - \|\theta^{(i+1)} - \theta^{ol}\|_2^2}{2\eta} + t\frac{\eta G^2}{2}.$$

$$\leq \frac{\|\theta^1 - \theta^{ol}\|_2^2}{2\eta} + \frac{\eta G^2}{2} \leq \frac{R^2}{2\eta} + \frac{t\eta G^2}{2}$$

$$= \frac{RG\sqrt{t}}{2} + \frac{RG\sqrt{t}}{2}$$

Recall: Stochastic gradient descent is an efficient offline optimization method, seeking $\hat{\theta}$ with

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Easily analyzed as a special case of online gradient descent!

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  - What does this imply about how Lipschitz f is?

$$\nabla f(\theta) = \sum_{j=1}^{n} \nabla f_j(\theta)$$

$$\|\nabla f(\theta)\| = \left\| \sum \nabla f_j(\theta) \right\|$$

$$\leq \sum_{j=1}^{n} \|\nabla f_j(\theta)\|$$

$$\leq n \cdot \frac{G}{n} \leq G.$$

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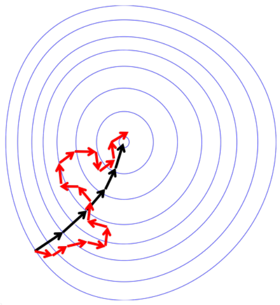
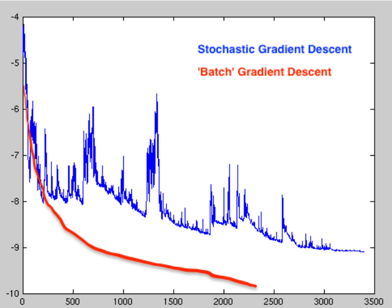
Stochastic Gradient Descent

- Set step size $\eta = \frac{R}{G\sqrt{t}}$.
- For $i = 1, \ldots, t$
  - Pick random $j_i \in 1, \ldots, n$.
  - $\vec{\theta}^{(i+1)} = \vec{\theta}^{(i)} - \eta \cdot \underline{\vec{\nabla} f_{j_i}}(\vec{\theta}^{(i)})$
- Return $\hat{\theta} = \frac{1}{t} \sum_{i=1}^{t} \vec{\theta}^{(i)}$.





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Note that: $\mathbb{E}[\vec{\nabla} f_{j_i}(\vec{\theta}^{(i)})] = \frac{1}{n} \vec{\nabla} f(\vec{\theta}^{(i)})$.

Analysis extends to any algorithm that takes the gradient step in expectation mini- (batch GD, randomly quantized, measurement noise, differentially private, etc.)

Theorem – SGD on Convex Lipschitz Functions: SGD run with $t \geq \frac{R^2G^2}{\epsilon^2}$ iterations, $\eta = \frac{R}{G\sqrt{t}}$, and starting point within radius R of θ^*, outputs $\hat{\theta}$ satisfying: $\mathbb{E}[f(\hat{\theta})] \leq f(\theta^*) + \epsilon.$

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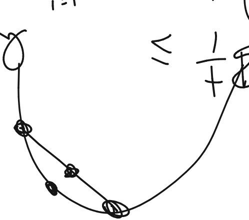
Step 1: $f(\hat{\theta}) - f(\theta^*) \leq \frac{1}{t}\sum_{i=1}^{t}[f(\theta^{(i)}) - f(\theta^*)]$

$$\hat{\theta} = \frac{1}{t}\sum_{i=1}^{t}\theta^i \Rightarrow f\left(\frac{1}{t}\sum\theta^i\right) - f(\theta^*)$$

Convexity

$$\leq \frac{1}{t}\sum\left[f(\theta^i) - f(\theta^*)\right]$$



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Step 2: $\mathbb{E}[f(\hat{\theta}) - f(\theta^*)] \leq \frac{n}{t} \cdot \mathbb{E}\left[\sum_{i=1}^{t}[f_{j_i}(\theta^{(i)}) - f_{j_i}(\theta^*)]\right]$.

$j_1, j_2, \ldots, j_{t-1}$

$$\mathbb{E}[f(\hat{\theta}) - f(\theta^*)] \leq \frac{1}{t}\sum_{i=1}^{t}\mathbb{E}[f(\theta^i) - f(\theta^*)]$$

$$= \frac{1}{t}\sum_{i=1}^{t}\mathbb{E}\, f_{j_i}(\theta^i)\cdot n - f_{j_i}(\theta^*)\cdot n]$$

$$f(\theta^i) = \sum_{j=1}^{n} f_j(\theta^i)$$

$$\mathbb{E}\, f_{j_i}(\theta^i) = \sum_{j=1} \Pr(j_i = j)\cdot f_j(\theta^i) = \frac{1}{n}\sum f_j(\theta^i)$$

$$= \frac{1}{n} f(\theta^i)$$

Theorem – SGD on Convex Lipschitz Functions: SGD run with $t \geq \frac{R^2G^2}{\epsilon^2}$ iterations, $\eta = \frac{R}{G\sqrt{t}}$, and starting point within radius R of θ^*, outputs $\hat{\theta}$ satisfying: $\mathbb{E}[f(\hat{\theta})] \leq f(\theta^*) + \epsilon$.

Step 1: $f(\hat{\theta}) - f(\theta^*) \leq \frac{1}{t}\sum_{i=1}^{t}[f(\theta^{(i)}) - f(\theta^*)]$

Step 2: $\mathbb{E}[f(\hat{\theta}) - f(\theta^*)] \leq \frac{n}{t} \cdot \mathbb{E}\left[\sum_{i=1}^{t}[f_{j_i}(\theta^{(i)}) - f_{j_i}(\theta^*)]\right].$

$\theta^{ol} = \arg\min_{\theta} \sum f_{j_i}(\theta)$

Step 3: $\mathbb{E}[f(\hat{\theta}) - f(\theta^*)] \leq \frac{n}{t} \cdot \mathbb{E}\left[\sum_{i=1}^{t}[f_{j_i}(\theta^{(i)}) - f_{j_i}(\theta^{ol})]\right].$

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$RG\sqrt{t}$

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$f_{j_1} \; f_{j_2} \ldots f_{j_t}$

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Step 4: $\mathbb{E}[f(\hat{\theta}) - f(\theta^*)] \leq \frac{n}{t} \cdot \underbrace{R \cdot \frac{G}{n} \cdot \sqrt{t}}_{\text{OGD bound}} = \frac{RG}{\sqrt{t}}.$

$\frac{RG}{\sqrt{t}} = \frac{RG}{RG/\epsilon} = \epsilon$

$\sum_{i=1}^{t} f_{j_i}(\theta^*) \geq \sum_{i=1}^{t} f_{j_i}(\theta^{ol})$

Stochastic gradient descent generally makes more iterations than gradient descent.

Each iteration is much cheaper (by a factor of n).

$$\vec{\nabla} \sum_{j=1}^{n} f_j(\vec{\theta}) \text{ vs. } \vec{\nabla} f_j(\vec{\theta})$$

When $f(\vec{\theta}) = \sum_{j=1}^{n} f_j(\vec{\theta})$ and $\|\vec{\nabla} f_j(\vec{\theta})\|_2 \leq \frac{G}{n}$:

Theorem – SGD: After $t \geq \frac{R^2 G^2}{\epsilon^2}$ iterations outputs $\hat{\theta}$ satisfying:

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When $\|\vec{\nabla} f(\vec{\theta})\|_2 \leq \bar{G}$:

Theorem – GD: After $t \geq \frac{R^2 \bar{G}^2}{\epsilon^2}$ iterations outputs $\hat{\theta}$ satisfying:

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When would this bound be tight?

Questions?