COMPSCI 690T Coding Theory and Applications — Apr. 18, 2017

Lecture 20

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1 How to build a finite field extension (in the context of Assignments)

1. Start with a finite field with prime number q of elements. For example, in a 3-ary finite field, $F_3 = \{0, 1, 2\}$.

2. Find an irreducible polynomial of degree t if you want a tth degree extension (to construct a field of size q^t). In F_3 an example is $p(x) = x^2 + 2x + 2$.

3. Let α be the root of $p(x)$. This implies that $x^2 + 2x + 2 = 0$, or equivalently, $\alpha^2 = -2\alpha - 2 = \alpha + 1$ in the above example.

4. 0 and powers of α generates the field. For example, $\{0, 1, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6, \alpha^7\}$ is F_9. So there is a polynomial representation corresponding to each exponential representation.

2 Fano's inequality (continuing from last class)

Theorem Let S be a random variable with finite sample space $|\Omega| = M$. Let X be another random variable that is obtained from S (such as S transmitted through some channel). $\hat{S}$ is the estimator for S that we derive from X with a function f such that $\hat{S} = f(X)$.

$$S \to X \to \hat{S}$$

Under this setup, the probability of error in estimating S is lower bounded by:

$$P_e = P(\hat{S} \neq S) \leq \frac{H(S|\hat{S}) - 1}{\log(M-1)})$$

3 Wiretap Channel

The wiretap channel is an information-theoretic model for communication in the presence of an eavesdropper. Suppose Alice wishes to communicate a message S to Bob. However, Eve can eavesdrop on the channel between Alice and Bob, and Alice wishes to make sure that Eve cannot decode (or only partly decode) the message S. A discrete-memoryless wiretap channel is a broadcast channel with sender Alice, legitimate receiver Bob and eavesdropper Eve. Alice wishes to communicate at a rate R to Bob while ensuring a given level equivocation for Eve.

$$K\text{-bit } S \xrightarrow{\text{encode}} N\text{-bit } X \xrightarrow{\text{decode}} \hat{S}$$

Alice — Eve sees μ-bit Z — Bob

Define:

$$P_e = P(S \neq \hat{S}) \qquad \Delta = H(S|Z) \qquad R = Rate = \frac{K}{N}$$

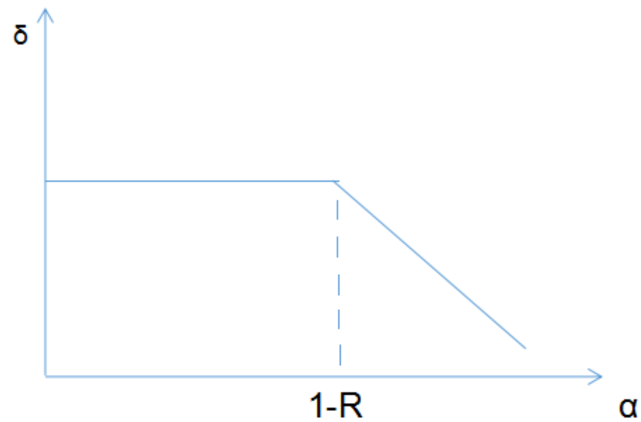
$$\alpha = \frac{\mu}{N} \qquad \delta = \frac{\Delta}{N}$$

We are going to prove

$$\Delta = \begin{cases} K \quad when \quad \mu \le N-K \\ KP_e + N - \mu \quad when \quad \mu \ge N-K \end{cases}$$

i.e.

$$\delta = \frac{\Delta}{K} \le \begin{cases} 1 \quad when \quad \alpha \le 1-R \\ P_e + \frac{1-\alpha}{R} \quad when \quad \mu \ge N-K \end{cases}$$



Proof:

$$\begin{aligned} \Delta &= H(S|Z) \\ &= H(S,Z) - H(Z) \\ &= H(S,X,Z) - H(X|S,Z) - H(Z) \\ &= H(S|X,Z) + H(X,Z) - H(Z) \\ &= H(S|X,Z,\hat{S}) + H(X|Z) \end{aligned}$$

$$\begin{aligned} \Delta &\le H(S|\hat{S}) + H(X|Z) \\ &= P_e \log(2^K - 1) + N - \mu \qquad (applied \quad Fano's \quad inequality) \end{aligned}$$

$$\begin{aligned} \tfrac{\Delta}{K} &\le P_e + \tfrac{N(1-\frac{\mu}{N})}{K} \\ &= P_e + \tfrac{1-\alpha}{R}. \end{aligned}$$

4 Data Compression

Suppose we have n bit sequence X.

$$X \in \{0,1\}^n.$$

We want to compress it into k bit sequence Y

$$Y \in \{0,1\}^k.$$

We want to recover the n bit sequence $\hat{X}$ from the k bit sequence Y within allowed error

$$X \to Y \to \hat{X}$$

We could have 2^k different $\hat{X}$. Let's call that set which form a ensemble $\mathcal{M}$, $|\mathcal{M}| = 2^k$.

Covering Code: A code $\mathcal{M} \subseteq \{0,1\}^n$ is called a covering code with covering radius δn if $\forall x \in \{0,1\}^n, \quad \exists \hat{x} \in \mathcal{M}$ such that $d(x, \hat{x}) \leq \delta n$ where $d(\cdot, \cdot)$ is the Hamming distance.

Let us show a lower bound for the size of a covering code with covering radius δn. Let us define the rate of a covering code to be $R \equiv \lim_n \frac{\log |\mathcal{M}|}{n}$.

We must have

$$|\mathcal{M}| \sum_{i=0}^{\delta n} \binom{n}{i} \geq 2^n$$

which gives us,

$$R \geq 1 - h(\delta).$$