CMPSCI 240: "Reasoning Under Uncertainty"

Lecture 24

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Reminders

- Check the course website: `http://www.cs.umass.edu/~wallach/courses/s12/cmpsci240/`
- Eighth homework is due on Friday

Grades

1. Add discussion section scores, divide by 14, multiply by 10
2. Add homework scores, divide by 250, multiply by 30
3. Divide midterm score by 100, multiply by 30
4. Add 1–3 to obtain your score (max. possible is 70)
5. If your score is 40 or less, you're in danger of getting a D

Recap

Last Time: Matrices and Vectors for Markov Chains

- Transition probability matrix:

$$A = \begin{pmatrix} p_{11} & p_{12} & p_{13} & \dots & p_{1S} \\ p_{21} & p_{21} & p_{22} & \dots & p_{2S} \\ \dots & \dots & \dots & \dots & \dots \\ p_{S1} & p_{S2} & p_{S3} & \dots & p_{SS} \end{pmatrix}$$

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- State probability vector: $v^{(t)}$ with $v_i^{(t)} = P(X_t = i)$

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- State probability vector: $v^{(t)}$ with $v_i^{(t)} = P(X_t = i)$
- *n*-step transition probabilities: $v^{(n)} = v^{(0)} A^n$

Last Time: Steady State

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- "Most" Markov chains have a unique steady state distribution that is approached by successive time steps (applications of transition matrix A) from any starting distribution
- $vA = v$ defines a set of $n + 1$ simultaneous equations

Examples of Steady State

- e.g., draw the transition probability graph and find the steady state distribution for a Markov chain with 4 states and

$$A = \begin{pmatrix} 0 & 0.9 & 0.1 & 0 \\ 0.2 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0.6 & 0.4 \end{pmatrix}$$

Irreducible Markov Chains

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- The steady state distribution is $v = \langle 0,\ 0,\ 0.375,\ 0.625 \rangle$
- There is no path from states 3 or 4 to states 1 or 2
- A Markov chain is irreducible if there exists a path in the transition graph from every state to every other state
- If a Markov chain is not irreducible, then it is reducible

Communicating Classes

- State i communicates with state j, i.e., $i \leftrightarrow j$, if there is some way of reaching state j from state i and vice versa

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- C is closed if the probability of leaving it is zero
- A Markov chain is irreducible if its state space forms a single communicating class, i.e., $i, j \in C$ for all $i, j \in \mathcal{S}$

Examples of Communicating Classes

- e.g., how many communicating classes are there if

$$A = \begin{pmatrix} 0 & 0.9 & 0.1 & 0 \\ 0.2 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0.6 & 0.4 \end{pmatrix}$$

Examples of Communicating Classes

- e.g., how many communicating classes are there if

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- e.g., what does this say about the reducibility of the chain?

Examples of Steady State

- e.g., draw the transition probability graph and find the steady state distribution for a Markov chain with 4 states and

$$A = \begin{pmatrix} 0 & 0.5 & 0 & 0.5 \\ 0.75 & 0 & 0.25 & 0 \\ 0 & 0.75 & 0 & 0.25 \\ 0.75 & 0 & 0.25 & 0 \end{pmatrix}$$

Examples of Steady State

- e.g., draw the transition probability graph and find the steady state distribution for a Markov chain with 4 states and

$$A = \begin{pmatrix} 0 & 0.5 & 0 & 0.5 \\ 0.75 & 0 & 0.25 & 0 \\ 0 & 0.75 & 0 & 0.25 \\ 0.75 & 0 & 0.25 & 0 \end{pmatrix}$$

- e.g., is this Markov chain irreducible?

Periodic Markov Chains

$$v^{(0)} = \langle 0.250, 0.250, 0.250, 0.250 \rangle$$

Periodic Markov Chains

$$\begin{aligned} v^{(0)} &= \langle 0.250, 0.250, 0.250, 0.250 \rangle \\ v^{(1)} &= \langle 0.375, 0.312, 0.125, 0.188 \rangle \end{aligned}$$

Periodic Markov Chains

$$
\begin{aligned}
v^{(0)} &= \langle 0.250, 0.250, 0.250, 0.250 \rangle \\
v^{(1)} &= \langle 0.375, 0.312, 0.125, 0.188 \rangle \\
v^{(2)} &= \langle 0.375, 0.281, 0.125, 0.219 \rangle
\end{aligned}
$$

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\vdots \quad & \vdots \quad \vdots
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$$

Periodic Markov Chains

$$v^{(0)} = \langle 1.000, 0.000, 0.000, 0.000 \rangle$$

Periodic Markov Chains

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v^{(1)} &= \langle 0.000, 0.500, 0.000, 0.500 \rangle
\end{aligned}
$$

Periodic Markov Chains

$$
\begin{aligned}
v^{(0)} &= \langle 1.000, 0.000, 0.000, 0.000 \rangle \\
v^{(1)} &= \langle 0.000, 0.500, 0.000, 0.500 \rangle \\
v^{(2)} &= \langle 0.750, 0.000, 0.250, 0.000 \rangle
\end{aligned}
$$

Periodic Markov Chains

$$
\begin{aligned}
v^{(0)} &= \langle 1.000, 0.000, 0.000, 0.000 \rangle \\
v^{(1)} &= \langle 0.000, 0.500, 0.000, 0.500 \rangle \\
v^{(2)} &= \langle 0.750, 0.000, 0.250, 0.000 \rangle \\
v^{(3)} &= \langle 0.000, 0.562, 0.000, 0.438 \rangle
\end{aligned}
$$

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v^{(6)} &= \langle 0.750, 0.000, 0.250, 0.000 \rangle
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v^{(6)} &= \langle 0.750, 0.000, 0.250, 0.000 \rangle \\
v^{(7)} &= \langle 0.000, 0.562, 0.000, 0.438 \rangle \\
v^{(8)} &= \langle 0.750, 0.000, 0.250, 0.000 \rangle \\
\vdots \quad & \vdots \quad \vdots
\end{aligned}
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- Chain is always in 1 or 3 on even t and 2 or 4 on odd t
- $X = \{1, 3\} \rightarrow Y = \{2, 4\}$ and vice versa
- If the chain is in a state in X at time t, then at time $t + 2$ it must return to a state in X; the same is true for Y

Periodic Markov Chains

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- A state i is periodic if the probability of returning to i is zero except at regular intervals (e.g., every 2 time steps)
- If all states are periodic, then the chain is periodic
- An irreducible Markov chain is periodic if there is some $k > 1$ such that A^k is the transition matrix of a reducible chain

Steady State

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- Reducible: some states can't be reached from others

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- Periodic: chain moves from group to group in a periodic way
- Reducible: some states can't be reached from others
- Steady state: if a Markov chain is aperiodic and irreducible then there must be some v such that for some t

$$|P(X_t = i) - v_i| \leq \epsilon$$

for any starting distribution and any positive, real ϵ

For Next Time

- Check the course website: `http://www.cs.umass.edu/~wallach/courses/s12/cmpsci240/`
- Eighth homework is due on Friday