CMPSCI 240: “Reasoning Under Uncertainty”

Lecture 5

Prof. Hanna Wallach
wallach@cs.umass.edu

February 7, 2012

Reminders

- Pick up a copy of B&T
- Check the course website: `http://www.cs.umass.edu/~wallach/courses/s12/cmpsci240/`
- First homework is due on Friday

Recap

Last Time: Counting

- Sampling k of n objects with replacement: n^k

Last Time: Counting

- Sampling k of n objects with replacement: n^k
- Permutations of n objects: $n!$

Last Time: Counting

- Sampling k of n objects with replacement: n^k
- Permutations of n objects: $n!$
- k-permutations of k of n objects: $\frac{n!}{(n-k)!}$

Last Time: Counting

- Sampling k of n objects with replacement: n^k
- Permutations of n objects: $n!$
- k-permutations of k of n objects: $\frac{n!}{(n-k)!}$
- Combinations of k of n objects: $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$

Last Time: Counting

- Sampling k of n objects with replacement: n^k
- Permutations of n objects: $n!$
- k-permutations of k of n objects: $\frac{n!}{(n-k)!}$
- Combinations of k of n objects: $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$
- Partitions of n objects into r subsets: $\frac{n!}{n_1!\,n_2!\ldots n_r!}$
- Challenge: try to show this yourself using a r-stage process!

Less Exciting Examples of Combinations

- e.g., a system contains $2x$ disks divided into x pairs, where each pair of disks contains the same data. If one of the disks in a pair fails, the data can be recovered, but if both disks fail, it cannot. Suppose two random disks fail. What is the probability that some data is inaccessible?

Binomial Probabilities

Sequences of Independent Trials

- Suppose we have a biased coin that lands heads with probability p. If we flip this coin 5 times, what is the probability that the outcome is $HHTTH$?

Sequences of Independent Trials

- Suppose we have a biased coin that lands heads with probability p. If we flip this coin 5 times, what is the probability that the outcome is $HHTTH$?
- If we flip the coin 5 times, what is the probability that the outcome consists of 3 heads and 2 tails in any order?

Binomial Probabilities

- Consider a sequence of n independent trials, each with a "success" probability p. The probability of any particular sequence with k successes is $p^k (1-p)^{n-k}$

Binomial Probabilities

- Consider a sequence of n independent trials, each with a "success" probability p. The probability of any particular sequence with k successes is $p^k (1-p)^{n-k}$
- The probability of exactly k successful trials is

$$\binom{n}{k} p^k (1-p)^{n-k}$$

Examples of Sequences of Independent Trials

- A cell phone provider can handle up to r data requests at once. Assume that every minute, each of the provider's n customers makes a request with probability p, independent of the behavior of the other customers. What is the probability that exactly x customers will make a data request during a particular minute? What is the probability that $> r$ customers will make a data request during a particular minute?

Random Variables

Random Variables

- Random variable: a real-valued function of the outcome of an experiment... though not written as a function!

Random Variables

- Random variable: a real-valued function of the outcome of an experiment... though not written as a function!
- e.g., rolling two dice, $\Omega = \{(1,1),(1,2),(1,3),\ldots,(6,6)\}$: examples of random variables include the sum of the two rolls, the number of 6s rolled, the product of the rolls, etc.

Random Variables

- Random variable: a real-valued function of the outcome of an experiment... though not written as a function!
- e.g., rolling two dice, $\Omega = \{(1,1),(1,2),(1,3),\ldots,(6,6)\}$: examples of random variables include the sum of the two rolls, the number of 6s rolled, the product of the rolls, etc.
- A random variable can map several outcomes to one value

Examples of Random Variables

- e.g., suppose we have 4 disks, each of which fails with probability p, $\Omega = \{0000, 0001, 0010, \ldots, 1111\}$: examples of random variables include the number of disks that failed, a boolean value indicating whether disks 1 or 2 failed, a boolean value indicating whether 3 or more disks failed, etc.

Random Variables and Events

- Uppercase letters, e.g., X or Y, denote random variables, while lowercase letters, e.g., x or y, denote their values

Random Variables and Events

- Uppercase letters, e.g., X or Y, denote random variables, while lowercase letters, e.g., x or y, denote their values
- Event $\{X = x\}$ is the event consisting of all outcomes in Ω that are mapped to value x by random variable X

Random Variables and Events

- Uppercase letters, e.g., X or Y, denote random variables, while lowercase letters, e.g., x or y, denote their values

- Event $\{X = x\}$ is the event consisting of all outcomes in Ω that are mapped to value x by random variable X
- e.g., flipping two coins, $\Omega = \{HH, HT, TH, TT\}$: if X is the number of heads flipped, what is the event $\{X = 1\}$?

Discrete vs. Continuous Random Variables

- Discrete random variable: the set of values is finite or countably infinite, e.g., rolling two dice: the number of 6s rolled, the sum of the rolls, the maximum roll, etc.

Discrete vs. Continuous Random Variables

- Discrete random variable: the set of values is finite or countably infinite, e.g., rolling two dice: the number of 6s rolled, the sum of the rolls, the maximum roll, etc.
- Continuous random variable: the set of values is uncountably infinite, e.g., choosing a in $[-1, 1]$: a^2, $a\,/\,2$, $a + 3.1$, etc.

Probability Mass Functions

Probability Mass Functions

- The probability mass function (PMF) of X is denoted by p_X

Probability Mass Functions

- The probability mass function (PMF) of X is denoted by p_X
- $p_X(x)$ is the probability of the event $\{X = x\}$:

$$p_X(x) = P(X = x) = P(\{X = x\})$$

Probability Mass Functions

- The probability mass function (PMF) of X is denoted by p_X
- $p_X(x)$ is the probability of the event $\{X = x\}$:

$$p_X(x) = P(X = x) = P(\{X = x\})$$

- e.g., flipping two (fair) coins, $\Omega = \{HH, HT, TH, TT\}$: if X is the number of heads flipped, what is $p_X(1)$?

Probability Mass Functions

- If $\{X \in S\}$ is the event consisting of all outcomes in Ω that are mapped to values $x \in S$ by random variable X, then

$$P(X \in S) = \sum_{x \in S} p_X(x) = \sum_{x \in S} P(\{X = x\})$$

Probability Mass Functions

- If $\{X \in S\}$ is the event consisting of all outcomes in Ω that are mapped to values $x \in S$ by random variable X, then

$$P(X \in S) = \sum_{x \in S} p_X(x) = \sum_{x \in S} P(\{X = x\})$$

- e.g., flipping two (fair) coins: $\Omega = \{HH, HT, TH, TT\}$: if X is the number of heads flipped, what is $P(X \geq 1)$?

Probability Mass Functions

- If $\{X \in S\}$ is the event consisting of all outcomes in Ω that are mapped to values $x \in S$ by random variable X, then

$$P(X \in S) = \sum_{x \in S} p_X(x) = \sum_{x \in S} P(\{X = x\})$$

- e.g., flipping two (fair) coins: $\Omega = \{HH, HT, TH, TT\}$: if X is the number of heads flipped, what is $P(X \geq 1)$?
- e.g., what is $\sum_x p_X(x)$ for any discrete random variable X?

Functions of Random Variables

- If X is a random variable and $g(\cdot)$ is some linear or nonlinear function, then $Y = g(X)$ is another random variable

Functions of Random Variables

- If X is a random variable and $g(\cdot)$ is some linear or nonlinear function, then $Y = g(X)$ is another random variable
- If X is discrete then Y is discrete with PMF

$$p_Y(y) = \sum_{\{x \mid g(x) = y\}} p_X(x)$$

Functions of Random Variables

- If X is a random variable and $g(\cdot)$ is some linear or nonlinear function, then $Y = g(X)$ is another random variable
- If X is discrete then Y is discrete with PMF

$$p_Y(y) = \sum_{\{x \mid g(x)=y\}} p_X(x)$$

- e.g., $Y = |X|$ and $p_X(x) = 1\,/\,9$ if x is an integer in the range $[-4, 4]$ and 0 otherwise: what do p_X and p_Y look like?

Common Discrete Random Variables

Discrete Uniform Random Variables

- A discrete uniform random variable X with range $[a, b]$ takes on any integer value between a and b inclusive

Discrete Uniform Random Variables

- A discrete uniform random variable X with range $[a, b]$ takes on any integer value between a and b inclusive
- The PMF of a discrete uniform random variable X is

$$p_X(k) = \frac{1}{b - a + 1} \text{ for } k = a, a + 1, \ldots, b$$

Discrete Uniform Random Variables

- A discrete uniform random variable X with range $[a, b]$ takes on any integer value between a and b inclusive
- The PMF of a discrete uniform random variable X is

$$p_X(k) = \frac{1}{b - a + 1} \text{ for } k = a, a + 1, \ldots, b$$

- e.g., Used to model probabilistic situations with $b - a + 1$ equally likely outcomes $(a, a + 1, \ldots, b)$, e.g., rolling a die

Bernoulli Random Variables

- Imagine a biased coin that lands heads (success) with probability p. X is a Bernoulli random variable if X is equal to 1 if the coin is heads (success) and 0 if it is tails (failure)

Bernoulli Random Variables

- Imagine a biased coin that lands heads (success) with probability p. X is a Bernoulli random variable if X is equal to 1 if the coin is heads (success) and 0 if it is tails (failure)
- The PMF of a Bernoulli random variable X is

$$p_X(k) = p \text{ if } k=1 \text{ and } p_X(k) = (1-p) \text{ if } k=0$$

Bernoulli Random Variables

- Imagine a biased coin that lands heads (success) with probability p. X is a Bernoulli random variable if X is equal to 1 if the coin is heads (success) and 0 if it is tails (failure)
- The PMF of a Bernoulli random variable X is

$$p_X(k) = p \text{ if } k = 1 \text{ and } p_X(k) = (1 - p) \text{ if } k = 0$$

- Used to model probabilistic situations with two outcomes, e.g., whether a server is online or an email is spam, etc.

Binomial Random Variables

- Suppose we flip a biased coin n times. X is a binomial random variable if X is equal to the number of heads (successes)

Binomial Random Variables

- Suppose we flip a biased coin n times. X is a binomial random variable if X is equal to the number of heads (successes)
- The PMF of a binomial random variable X is

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k} \text{ for } k = 0, 1, 2, \ldots, n$$

Binomial Random Variables

- Suppose we flip a biased coin n times. X is a binomial random variable if X is equal to the number of heads (successes)
- The PMF of a binomial random variable X is

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k} \text{ for } k = 0, 1, 2, \ldots, n$$

- X is the sum of n independent Bernoulli random variables, e.g., number of servers (out of n) that are online, etc.

Examples of Binomial Random Variables

- e.g., you go to a party with 500 guests. What is the probability that one other guest has the same birthday as you?

[Got to here in class...]

Geometric Random Variables

- Suppose we flip a biased coin repeatedly until it lands heads (success). X is a geometric random variable if X is equal to the number of times that we have to flip the coin

Geometric Random Variables

- Suppose we flip a biased coin repeatedly until it lands heads (success). X is a geometric random variable if X is equal to the number of times that we have to flip the coin
- The PMF of a geometric random variable X is

$$p_X(k) = (1-p)^{k-1}\, p \text{ for } k = 1, 2, 3, \ldots$$

Geometric Random Variables

- Suppose we flip a biased coin repeatedly until it lands heads (success). X is a geometric random variable if X is equal to the number of times that we have to flip the coin
- The PMF of a geometric random variable X is

$$p_X(k) = (1-p)^{k-1}\, p \text{ for } k = 1, 2, 3, \ldots$$

- Used to model the number of repeated independent trials up to (and including) the first "successful" trial, e.g., spam

Examples of Geometric Random Variables

- e.g., you have 5 keys for your new apartment but you don't know which key opens the front door. What is the PMF of the number of trials needed to open the front door assuming that at each trial you are equally likely to choose any of the 5 keys?

Poisson Random Variables

- The PMF of a Poisson random variable X is

$$p_X(k) = e^{-\lambda} \frac{\lambda^k}{k!} \text{ for } k = 0, 1, 2, \ldots$$

Poisson Random Variables

- The PMF of a Poisson random variable X is

$$p_X(k) = e^{-\lambda} \frac{\lambda^k}{k!} \text{ for } k = 0, 1, 2, \ldots$$

- A Poisson PMF with $\lambda = np$ is a good approximation for a binomial PMF with very small p and very large n if $k \ll n$

Poisson Random Variables

- The PMF of a Poisson random variable X is

$$p_X(k) = e^{-\lambda} \frac{\lambda^k}{k!} \text{ for } k = 0, 1, 2, \ldots$$

- A Poisson PMF with $\lambda = np$ is a good approximation for a binomial PMF with very small p and very large n if $k \ll n$
- e.g., the number of typos in a book with n words, number of cars (out of n) that crash in a city on a given day, etc.

Examples of Poisson Random Variables

- e.g., you go to a party with 500 guests. What is the probability that one other guest has the same birthday as you?

For Next Time

- Read B&T 2.4, 2.5
- Check the course website: `http://www.cs.umass.edu/~wallach/courses/s12/cmpsci240/`
- First homework is due on Friday