CMPSCI 240: "Reasoning Under Uncertainty"

Lecture 4

Prof. Hanna Wallach
wallach@cs.umass.edu

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Reminders

- Pick up a copy of B&T
- Check the course website: `http://www.cs.umass.edu/~wallach/courses/s12/cmpsci240/`
- First homework will be assigned tomorrow

Recap

Last Time: Total Probability Theorem

- If $A_1, \ldots, A_n$ partition Ω then for any event B

$$\begin{aligned} P(B) &= P(B \cap A_1) + \ldots + P(B \cap A_n) \\ &= \sum_{i=1}^{n} P(A_i)\, P(B \mid A_i) \end{aligned}$$

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- "Divide-and-conquer" approach to finding $P(B)$

Last Time: Bayes' Rule

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- Useful for inference, i.e., where we know $P(B \mid A_i)$ and $P(A_i)$ for every i and want to find $P(A_i \mid B)$ for some i

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- e.g., flipping two coins, $H_1 =$ "first flip is heads" $= \{HH, HT\}$, $H_2 =$ is "second flip is heads" $= \{HH, TH\}$, and $D =$ "the flips are different" $= \{HT, TH\}$: H_1 and H_2 are independent, what about H_1 and D?

Independence of Multiple Events

- Events A, B, and C are independent if and only if

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- Pairwise independence does not imply independence

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- e.g., flipping two coins, $H_1 =$ "first flip is heads" $= \{HH, HT\}$, $H_2 =$ is "second flip is heads" $= \{HH, TH\}$, and $D =$ "the flips are different" $= \{HT, TH\}$: we already know that H_1 and H_2 are independent, but are they conditionally independent given D?

Counting

Equally Likely Outcomes

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- Discrete uniform probability law: If Ω is finite and all outcomes are equally likely, then $P(A) = |A| \,/\, |\Omega|$
- How can we count $|A|$ and $|\Omega|$?

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- Consider a process with r stages, e.g., rolling r dice
- There are n_1 possible choices at the first stage
- For each of these, there are n_2 possible choices at stage 2
- In general, for each possible choice at stage $i-1$, there are n_i possible choices at stage $i \implies$ the total number of choices (i.e., outcomes for the entire process) is $n_1\, n_2\, n_3 \ldots n_r$

Examples of the Counting Principle

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- e.g., if $A = \{x_1, \ldots, x_n\}$ how many subsets does A have?

Sampling with Replacement

- e.g., drawing $r = 5$ cards from a deck of $n = 52$ cards with replacement: $n_1 = n_2 = \ldots n_5 = n = 52$, so there are $n^r = 52^5$ ways of drawing 5 cards with replacement

Permutations

Permutations

- e.g., how many ways can we assign n threads to n processors, such that each thread is assigned to exactly one processor and each processor is assigned exactly one thread?

k-Permutations

- e.g., how many ways can we assign n threads to $k \leq n$ processors such that no thread is assigned to multiple processors and each processor is assigned exactly one thread?

Examples of Permutations

- e.g., suppose you have 4 books about competitive eating, 10 books about Linux, and 2 books about roller derby. How many ways can you arrange these books on a shelf such that all of the books on a given subject are grouped together?

Combinations

When Order Doesn't Matter

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- When order doesn't matter some permutations are indistinguishable from others, e.g., pizza toppings: bacon, ham, and sausage vs. sausage, bacon, and ham

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- When order doesn't matter, the number of ways to choose k elements from a set of n elements (i.e., combinations) is

$$\frac{\#\ k\text{-permutations}}{k!} = \frac{n!}{k!\,(n-k)!} = \binom{n}{k}$$

Examples of Combinations

- e.g., Antonios offers fifteen different toppings. How many ways can you create a pizza with three distinct toppings?

Real-World Examples of Combinations

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- Last year, CNN Money decided to verify this claim...

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- 12 sauces (mustard, mayo, ketchup, ...): can have 0–3
- 7 cheeses (blue, goat, cheddar, ...): can have 0–2
- 8 patties (beef, pork, egg, lamb, ...): can have 1

Less Exciting Examples of Combinations

- e.g., a system contains $2x$ disks divided into x pairs, where each pair of disks contains the same data. If one of the disks in a pair fails, the data can be recovered, but if both disks fail, it cannot. Suppose two random disks fail. What is the probability that some data is inaccessible?

[Got to here in class...]

Binomial Probabilities

Sequences of Independent Trials

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- If we flip the coin 5 times, what is the probability that the outcome consists of 3 heads and 2 tails in any order?

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- Consider a sequence of n independent trials, each with a "success" probability p. The probability of any particular sequence with k successes is $p^k\,(1-p)^{n-k}$

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- Consider a sequence of n independent trials, each with a "success" probability p. The probability of any particular sequence with k successes is $p^k\,(1-p)^{n-k}$
- The probability of exactly k successful trials is

$$\binom{n}{k}\,p^k\,(1-p)^{n-k}$$

Examples of Sequences of Independent Trials

- A cell phone provider can handle up to r data requests at once. Assume that every minute, each of the provider's n customers makes a request with probability p, independent of the behavior of the other customers. What is the probability that exactly x customers will make a data request during a particular minute? What is the probability that $> r$ customers will make a data request during a particular minute?

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- Combination: k elements from a n element set, ignoring order
- A combination partitions the set in two: elements that belong to the k-element combination, and elements that don't
- What about partitioning n elements into r disjoint subsets of sizes $n_1, n_2, \ldots, n_r$? How many ways can we do this?

Partitions

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- Form the subsets one at a time using a r-stage process
- There are $\binom{n}{n_1}$ ways to form the first subset
- For each of these, there are $\binom{n-n_1}{n_2}$ ways to form subset 2
- In general, for each possible way to form subset $i-1$, there are $\binom{n-n_1-\ldots-n_{i-1}}{n_i}$ ways to form subset $i \implies$ there are $\frac{n!}{n_1!\,n_2!\ldots n_r!} = \binom{n}{n_1, n_2, \ldots, n_r}$ ways to partition the set

For Next Time

- Read B&T 2.1, 2.2, 2.3
- Check the course website: `http://www.cs.umass.edu/~wallach/courses/s12/cmpsci240/`
- First homework will be assigned tomorrow