CMPSCI 240
Reasoning Under Uncertainty
Homework 8

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Assigned: April 20, 2012
Due: April 27, 2012

Question 1: Suppose that you have a Markov chain with 3 states $\mathcal{S} = \{s_1, s_2, s_3\}$ and the following transition probabilities between states:

$$P(X_{t+1} = s_2 \mid X_t = s_1) = 2/5$$
$$P(X_{t+1} = s_3 \mid X_t = s_1) = 3/5$$
$$P(X_{t+1} = s_1 \mid X_t = s_2) = 4/7$$
$$P(X_{t+1} = s_2 \mid X_t = s_2) = 2/7$$
$$P(X_{t+1} = s_3 \mid X_t = s_2) = 1/7$$
$$P(X_{t+1} = s_2 \mid X_t = s_3) = 1/2$$
$$P(X_{t+1} = s_3 \mid X_t = s_3) = 1/2$$

(a) Draw the state transition diagram (transition probability graph) for this Markov chain, annotated with the transition probabilities.

(b) Find the steady state distribution for this Markov chain. Recall that the steady state distribution is the vector v that satisfies:

(1) $v = vA$ where A is the matrix of transition probabilities,

(2) $v_1, v_2, v_3 \geq 0$
(3) $v_1 + v_2 + v_3 = 1$.

Question 2: Consider a Markov chain with states $\mathcal{S} = \{s_1, s_2, s_3\}$. Suppose the chain is initially in state s_1 and the transition matrix is:

$$A = \begin{pmatrix} 2/5 & 2/5 & 1/5 \\ 0 & 1/2 & 1/2 \\ 3/4 & 0 & 1/4 \end{pmatrix}$$

i.e., $P(X_t = s_1 \,|\, X_{t-1} = s_1) = 2/5$, $P(X_t = s_2 \,|\, X_{t-1} = s_1) = 2/5$, etc.

(a) What is the probability that the chain is in s_1 after 2 steps?

(b) What is the steady state distribution of the chain?

(c) What is the probability that the chain is in s_1 after 4 and 8 steps? (Hint: you can compute A^{2k} by multiplying A^k and A^k together.)

Question 3: Suppose you generate a random bit sequence. The probability of generating a 0 is $2/3$. The probability of generating a 1 is $1/3$. Having generated a sequence, you count the number of 1s generated modulo 4.

(a) Draw the state transition diagram, annotated with the transition probabilities, for the Markov chain representing this scenario.

(b) Assume that the number of 1s generated is 0 at time $t = 0$ and find the probability of being in each of the states at time $t = 2$.

(c) Find the steady state distribution for this Markov chain.

Question 4: There are 3 stones. If a frog is on stone 1, the probability that it will jump to stone 2 is 0.2, the probability that it will remain on stone 1 is 0.1, and the probability that it will jump to stone 3 is 0.7. If the frog is on stone 2, the probability that it will jump to stone 1 is 0.2, the probability that it will jump to stone 3 is 0.6, and the probability that it will remain on stone 2 is 0.2. If the frog is on stone 3, the probability that it will jump to stone 1 is 0.6, the probability that it will remain on stone 3 is 0.3, and the probability that it will jump to stone 2 is 0.1. The frog is on stone 1 initially.

(a) Draw the state transition diagram and write down the transition probability matrix for the Markov chain that models the above scenario.

(b) What is the probability the frog is on stone 2 after two hops? You should calculate this probability using matrix multiplication.

(c) What is the probability that the frog is on stone 1 after three hops? You should calculate this n-step transition probability using your answer from part (b) and the law of total probability.