CMPSCI 105 – Lecture #2 Numbers and the Computer

Definition: BIT

- Binary Digit
- Smallest possible unit of information
- Two values only: 0 or 1
- Represent a single Yes or No question
- Can encode any two-valued system
  - Yes/No, True/False, Up/Down, On/Off, In/Out, etc.
- Easy to build hardware to encode bits.

Bits and Patterns

- 1 Bit gives 2^1 = 2 patterns: 0 or 1
- 2 Bits gives 2^2 = 4 patterns: 00, 01, 10, 11
- 3 Bits gives 2^3 = 8 patterns: 000, 001, 010, 011, 100, 101, 110, 111
- Each new bit doubles the number of patterns
- Therefore: N Bits gives 2^N Distinct patterns.

Definition: Byte

- Packet of 8 Bits (French word is "octet")
- Typical unit of computer memory / storage
- Used to represent one standard character
- Values range from 00000000 … 11111111
- 2^8=256 Distinct patterns
- Can encode any integer between 0 and 255

Unsigned Integers

- Pick storage size of N bits (8, 16, 32, 64, etc.)…
- …therefore 2^N distinct patterns are available.
- Smallest value is all zeroes (decimal value 0),
- Largest value is therefore 2^N-1.
- Results less than zero are "underflow" errors,
- Results greater than max are "overflow" errors
- Each computer architecture has a fixed N.

Signed Integers

- Pick N, there are still 2^N patterns.
- Consider half the patterns to be negative.
  - Half of $2^N = 2^N/2 = 2^{N-1}$.
- Remaining patterns are zero and above.
- Signed range is therefore $-2^{N-1} \ldots +2^{N-1}-1$.
  - Zero is considered positive.

Example for N=8

- 2^8 = 256 patterns
- Unsigned Range
  - Minimum: 0
  - Maximum: 2^8-1 = 255
- Signed Range
  - Minimum: -2^{8-1} = -2^7 = -128
  - Maximum: +2^{8-1}-1 = +2^7-1 = +128-1 = +127

Example for N=16

- 2^{16} = 65536 patterns
- Unsigned Range
  - Minimum: 0
  - Maximum: 2^{16}-1 = 65535
- Signed Range
  - Minimum: -2^{16-1} = -2^{15} = -32768
  - Maximum: +2^{16-1}-1 = +2^{15}-1 = +32768-1 = +32767

Example for N=32

- 2^{32} = 4,294,967,296 patterns
- Unsigned Range
  - Minimum: 0
  - Maximum: 2^{32}-1 = 4,294,967,295
- Signed Range
  - Minimum: -2^{32-1} = -2^{31} = -2,147,483,648
  - Maximum: +2^{32-1}-1 = +2^{31}-1 = +2,147,483,647
  - Nine (and a little more) significant digits

What about Real numbers?

- Approaches:
  - Rational
  - Fixed-Point
  - Floating-Point
- All require re-interpreting how bits are used.

Rational Numbers

- For N bits, divide into two $^N/_2$ bit sections:
  - First section is numerator
  - Second section is denominator
- Numbers like $^1/_3$, $^1/_2$, $^3/_7$, $^1/_{10}$, $^{355}/_{113}$ are easy
- Reduce to lowest form (e.g., $^2/_4$ goes to $^1/_2$)
- Many redundant patterns:
  - low information density
  - Not efficient use of bits

Fixed-Point Numbers

- Set virtual decimal point to middle of bits:
  - Half the bits are integer
  - Half the bits are fraction
- All bit patterns are useful
- Easy to add, subtract, multiply, divide in binary
- Trades off range of values for fraction support.
  - For N=16, max signed value is only +127.99609375
- Still not an efficient use of bits

Floating-Point Numbers

- Binary version of Scientific Notation
  - Decimal: $+3.4024 \times 10^{15}$
  - Binary: $+1.00101001 \times 2^{1001}$
- Use one bit for *sign* (0=plus, 1=minus)
- Use some of the N bits for *exponent*
- Use remaining bits for *mantissa* (significand)
- Trades off <u>precision</u> for <u>dynamic range</u>

Floating-Point Precision

- Single Precision
  - N=32 bits (1 sign, 8 exponent, 23 mantissa)
  - Dynamic Range: $\pm 10^{\pm 38}$
  - Significant Figures: 5-6 Decimal Digits
  - (Remember 32 bit integers have about 9 sig. figs.)
- Double Precision (Used by Excel)
  - N=64 bits (1 sign, 11 exponent, 52 mantissa)
  - Dynamic Range $\pm 10^{\pm 308}$
  - Significant Figures: 15-16 Decimal Digits

But long fractions get rounded off:

- Expected loss of precision:
  - Numbers with naturally long but finite fractions,
  - Rationals that repeat forever (⅓ = 0.333333333…),
  - Irrationals (e, π, ϕ, √2, √3, √5, etc.).
- Unexpected loss of precision: Well-behaved decimal fractions that are ill-behaved in binary ($^1/_{10}$ = 0.00011001100110011…)

Aside: Proof that √2 is Irrational

- √2 = 1.414213562…
- Remember:
  - Even × Even = Even (4 × 6 = 24)
  - Even × Odd = Even (4 × 7 = 28)
  - Odd × Odd = Odd (5 × 7 = 35)
- Assume √2 is Rational: $\sqrt{2} = {}^P/_Q$
- Assume Lowest Form: P, Q aren't <u>both</u> even
  - (if both were even, we can repeatedly divide both P and Q by 2 until at least one is odd)

Aside: Proof that √2 is Irrational

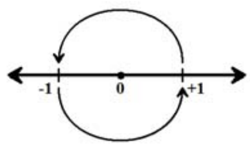
- Square both sides: $2 = P^2 / Q^2$
- Multiply by Q^2: $2Q^2 = P^2$
- Conclusion #1: P^2 is even, thus P is even
- Divide by 2: $Q^2 = P^2 / 2 = P \times {}^P/_2$
- Conclusion #2: Q^2 is even, thus Q is even
- Contradiction:
  - Initial assertion was P, Q <u>aren't both</u> even, proof says <u>both are</u> even, thus assumption that $\sqrt{2} = {}^P/_Q$ is false. No such rational number exists.

The Biggest Dirty Secret of Computing

- Most of the interesting numbers in the Universe are <u>irrational</u>,
- Numbers on computers have a <u>fixed and finite</u> number of bits,
- Therefore, most values get rounded off.
- **Most numerical results are approximations.**
- More bits means more precision, but only forestalls and does not eliminate the problem.

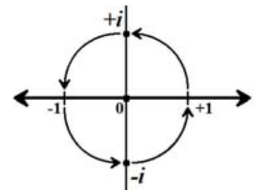
Complex Numbers

- The Real number line extends from -∞ to +∞,
- Use of space above and below the line gives us more computational expressive power.
- Negation then becomes a rotation of 180°:



Complex Numbers

- Rotation of +1 by 90° leaves it in space above the zero center. Call that number *i*:



Complex Numbers

- Multiplying a number by *i* twice equals negation,
- Thus $i^2 = -1$, and then $i = \sqrt{-1}$
- *i* is called "imaginary"

Complex Numbers

- A complex number is then a pair of numbers:
  - A value along the Real axis,
  - A value along the Imaginary axis.
  - Written with the Real part first, then Imaginary.
- Examples:
  - 2+3*i*, 5-7*i*, -3+2*i*, -4-6*i*, 6.7+5.9*i*, etc.
  - 7 (same as 7+0*i*)

Complex Math

- Add/Subtract: treat components separately:
  - $2+6i + 5-2i = (2+5) + (6-2)i = 7+4i$
  - $2+6i - 5-2i = (2-5) + (6--2)i = -3+8i$
- Multiplication uses FOIL method:
  - $2+6i \times 5-2i =$
  - $(2 \times 5) + (2 \times -2i) + (6i \times 5) + (6i \times -2i) =$
  - $(10) + (-4i) + (30i) + (-12i^2) =$
  - $(10+12) + (-4+30)i = 22+26i$

Complex Math

- Division uses two complex multiplications to eliminate imaginary component in denominator,
- Multiply both numerator and denominator by complex conjugate of denominator.
- Example:
  - $22+26i \div 2+6i =$
  - Numerator: $22+26i \times 2-6i = 200-80i$
  - Denominator: $2+6i \times 2-6i = 4+36 = 40$
  - $200-80i \div 40 = 5-2i$

Complex Math

- Used in math, engineering, physics, etc.
- Supported by early language FORTRAN,
- Supported by modern language Python,
- Supported (badly) by Excel 2007 and later.
- Mostly Double-Precision Floats,
- Subject to same round-off errors as other floating-point numbers.