COMPSCI 688: Probabilistic Graphical Models

Lecture 7: Undirected Graphical Models: Examples and Inference

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Review

Markov Random Fields

A Markov random is a distribution that factors over a set of "cliques" $\mathcal{C}$:

$$p(\mathbf{x}) = \frac{1}{Z}\prod_{c\in\mathcal{C}}\phi_c(\mathbf{x}_c), \quad Z = \sum_{\mathbf{x}}\prod_{c\in\mathcal{C}}\phi_c(\mathbf{x}_c)$$

The *dependence graph* $\mathcal{G} = (V, E)$ is the graph where nodes i and j are connected by an edge if they appear together in some factor.

We say that $p(\mathbf{x})$ *factors* over $\mathcal{G}$, and denote this property as (F).

Markov Properties

The *global Markov property* (G) connects conditional indpendence to graph separation.

Distribution $p(\mathbf{x})$ satisfies the global Markov property with respect to $\mathcal{G}$ if

$$\text{sep}_{\mathcal{G}}(A, B|S) \implies \mathbf{X}_A \perp \mathbf{X}_B \mid \mathbf{X}_S \qquad \text{(G)}$$

There are two other Markov properties (*local* and *pairwise*) implied by the global Markov property.

Factorization and Markov Properties

It's easy to show that factorization implies Markov: (F) ⇒ (G).

There is a famous partial converse. For a *positive* distribution: (G) ⇒ (F)

Theorem (Hammersley-Clifford). If $p(\mathbf{x}) > 0$ for all $\mathbf{x}$, then (F) $\iff$ (G)

Examples

Example: Ising Model

- $\mathcal{G}$ is a lattice and $X_i \in \{-1, 1\}$
- Have *unary potential* β_i for each node i and *pairwise potential* β_{ij} for each edge (i, j)

$$p(\mathbf{x}) = \frac{1}{Z}\prod_i \beta_i(x_i) \prod_{(i,j)\in E} \beta_{ij}(x_i, x_j)$$

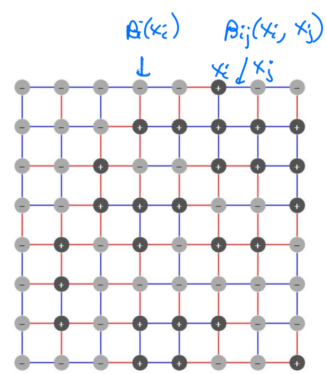
$$\beta_i(x_i) = \exp(b_i x_i)$$

$b_i = 10$; e^{10} $x_i = 1$, e^{-10} $x_i = -1$

$$\beta_{ij}(x_i, x_j) = \exp(b_{ij} x_i x_j)$$

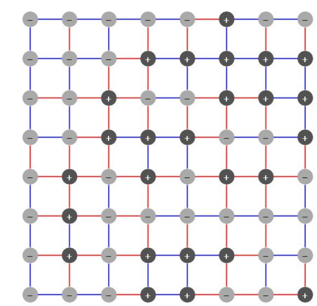
$b_{ij} = 10$; e^{10} $x_i = x_j$, e^{-10} $x_i \neq x_j$

- $b_i > 0 \implies X_i$ likes to be positive
- $b_{ij} > 0 \implies X_i$ and X_j like to be the same



Example: Ising Model

- In general, Markov networks can be seen as expressing preferences for certain local configurations of the variables.
- Joint configurations with high probability balance the preferences of all factors.



Example: Simulating an Ising Model

Demo: Ising Model

$$p(\mathbf{x}) = \frac{\exp\left(\frac{1}{T}\sum_{(i,j)\in E} x_i x_j\right)}{Z}$$

T = temperature > 0

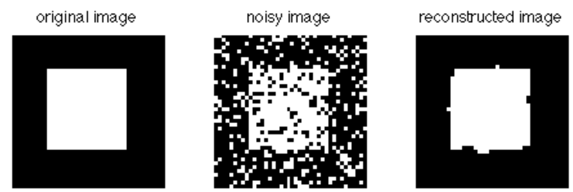
T high, weak preference for $x_i = x_j$

b_{ij}

$= \frac{\prod_{(i,j)\in E} \exp(b_{ij} x_i x_j)}{Z}$

Example: Statistical Image Models

The Ising model with $b_{ij} > 0$ prefers smoothness, and can be used as a model for images in denoising procedures:

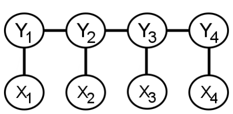


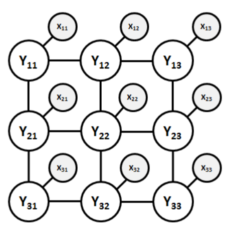
Example: Image Denoising

Example: Part-of-Speech Tagging

Conditional Random Fields

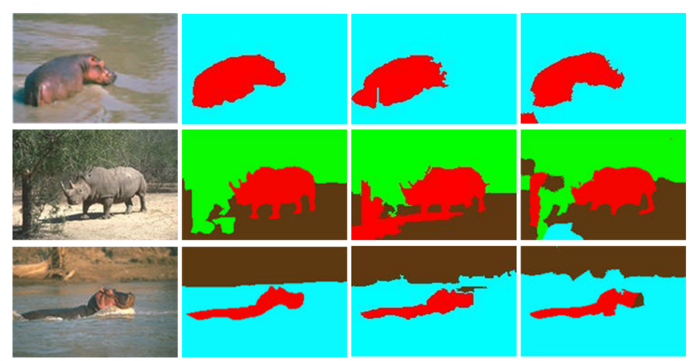
The previous two examples were examples of **conditional random fields** (CRFs), a very important model class in machine learning. A CRF is essentially a Markov network where one set of nodes is always conditioned on.



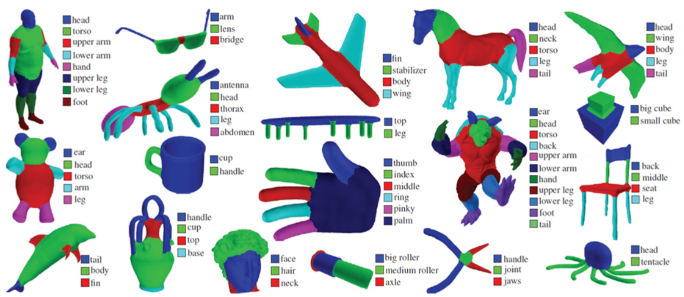


The $\mathbf{y}$ nodes are *labels*, and the $\mathbf{x}$ nodes are *features*.

Example: Image Segmentation



Example: 3D Mesh Segmentation



Example: Bayes Nets as MRFs

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Some structure is lost in this transformation. When we replace $p(a|b,c)$ by $\phi(a,b,c)$, we "forget" that a Bayes net is **locally normalized**

$$\sum_a \phi(a,b,c) = 1 \quad \forall b, c.$$

This is a special property of Bayes nets and is central to V-structures, explaining away, and D-separation. It occurs "internally" to the factor $\phi(a,b,c)$ and is not represented in the MRF graph structure.

Similarly, when we replace $\prod_i p(x_i|\mathbf{x}_{\mathsf{pa}(i)})$ by $\frac{1}{Z}\prod_{c\in\mathcal{C}}\phi_c(\mathbf{x}_c)$, we "forget" that a Bayes net is **globally normalized**:

$$\sum_x \prod_{c\in\mathcal{C}} \phi_c(\mathbf{x}_c) = 1 \implies Z = 1.$$

This is another special property of Bayes nets that makes learning easy.