

# COMPSCI 688: Probabilistic Graphical Models

## Lecture 4: Directed Graphical Models: D-Separation, Queries

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Review

Review

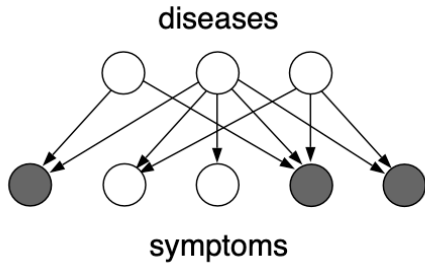
- ▶ Bayes net:  $p(\mathbf{x}) = \prod_{i=1}^N p(x_i | \mathbf{x}_{\text{pa}(i)})$
- ▶ Factorization  $\iff$  conditional independence (one statement per node)

$$p(\mathbf{x}) = \prod_{i=1}^N p(x_i | \mathbf{x}_{\text{pa}(i)}) \iff X_i \perp \mathbf{X}_{\text{nd}(i)} | \mathbf{X}_{\text{pa}(i)} \text{ for all } i$$

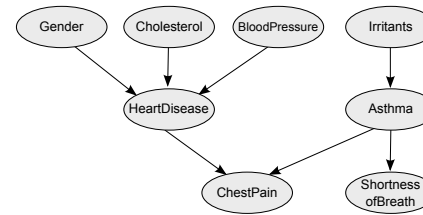
- ▶ We would like to chain together conditional independence properties using the graph structure to derive new ones  $\rightarrow$  D-separation

Examples

### Example 1: Medical Diagnosis

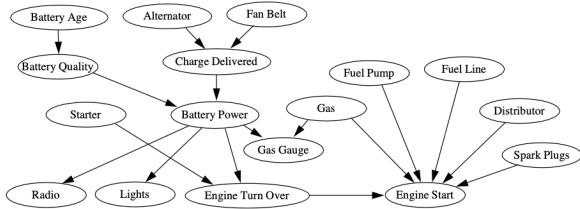


### Example 2: Medical Diagnosis

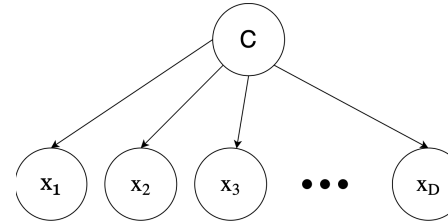


### Example 3: Equipment Diagnostics

Suppose engine does not start but radio plays? What do you believe about gas in tank?



### Example 4: Naive Bayes



- ▶ given class variable  $C$  (discrete), the observed features  $X_1, X_2, X_3, \dots, X_D$  are conditionally independent:  $\forall i \neq j, X_i \perp X_j | C$
- ▶ joint distribution is

$$p(c, x) = p(c) \prod_{i=1}^D p(x_i | c).$$

## Warning: Causality

### Bayes nets are not causal

- ▶ Many of our examples are motivated by a causal model, but Bayes net arrows could just as easily point from “effect” to “cause” (or have no causal semantics at all)
- ▶ Can be given causal semantics (beyond our scope)

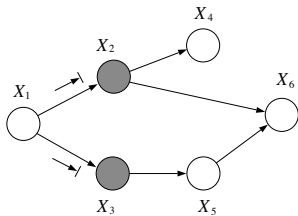
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## D-Separation

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## Independence Properties

So far, we know  $X_i \perp X_{nd(i)} | X_{pa(i)}$  for all  $i$



However, this also implies other conditional independence properties. E.g., it's true that  $X_1 \perp X_6 | X_2, X_3$  in this network. How can we determine this?

The core principles can be understood by examining three-node networks, then “chaining” ideas together...

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## Three-Node Bayes Nets: Common Parent, Chains

Networks  $A \leftarrow B \rightarrow C$ ,  $A \rightarrow B \rightarrow C$  and  $C \rightarrow B \rightarrow A$

$A \perp C$  but  $A \perp C | B$ . Observing  $B$  blocks dependence of  $A$  and  $C$

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## Three-Node Bayes Nets: V-Structure

Network  $A \rightarrow B \leftarrow C$

$A \perp C$  but  $A \not\perp C | B$ . Observing  $B$  induces dependence of  $A$  and  $C$

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## Explaining Away

- ▶ “Explaining away” via V-structures is a distinguishing property of Bayes nets:
- ▶ **Example:** Alice is late. Was she abducted by aliens or did she forget her watch?

In words: if there are two possible causes for the observed evidence, knowing about one of the causes provides information about the other

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## D-Separation

*Directed separation* or **D-separation** is a definition of separation in a directed graph that corresponds exactly to conditional independence in Bayes nets

A three-node path is blocked iff has one of the following types:

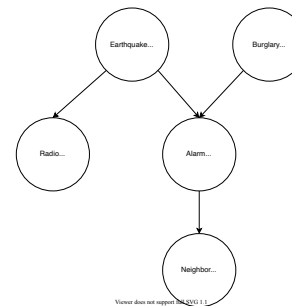
- 1)  $A \rightarrow B \rightarrow C$  or  $C \rightarrow B \rightarrow A$  and  $B$  is observed
- 2)  $A \leftarrow B \rightarrow C$  and  $B$  is observed
- 3)  $A \rightarrow B \leftarrow C$  and neither  $B$  nor any descendent of  $B$  is observed

Let  $\mathbf{X}$ ,  $\mathbf{Y}$ , and  $\mathbf{Z}$  be three sets of nodes.  $\mathbf{X}$  and  $\mathbf{Y}$  are d-separated given observed nodes  $\mathbf{Z}$  iff every path from  $\mathbf{X}$  to  $\mathbf{Y}$  is blocked, where a path is blocked if any three-node sequence in the path is blocked.

$\mathbf{X}$  and  $\mathbf{Y}$  are d-separated given  $\mathbf{Z} \iff \mathbf{X} \perp \mathbf{Y} | \mathbf{Z}$

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## Example: D-separation in the Alarm Model



$E \perp B | A, R?$

$E \perp B | R, N?$

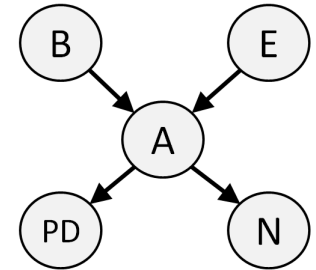
$E \perp B | R?$

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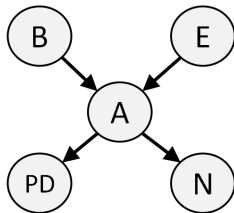
### Queries

### The Alarm Network (II)

- ▶ You live in the suburbs of LA. Your home alarm may go off because of a break-in or earthquake. If your alarm goes off you might get a call from the police or your neighbor.
- ▶ **Random Variables:** Break-in (B), Earthquake (E), Alarm (A), Police Department calls (PD), Neighbor calls (N).



### The Alarm Network: Factorization



▶ **Factorization:**  $P(B, E, A, PD, N) = P(B)P(E)P(A|B, E)P(PD|A)P(N|A)$

### The Alarm Network: Parameters

B	E	$P(A=1 B,E)$
1	1	0.950
1	0	0.940
0	1	0.290
0	0	0.001

$P(B=1)$
0.001

$P(E=1)$
0.002

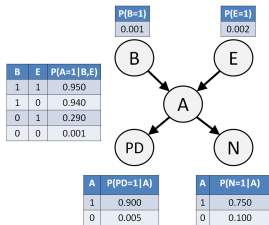
A	$P(PD=1 A)$
1	0.900
0	0.005

A	$P(N=1 A)$
1	0.750
0	0.100

## The Alarm Network: Joint Query

- **Question:** What is the probability that there is a break-in, but no earthquake, the alarm goes off, the police call, but your neighbor does not call?

$$\begin{aligned} P(B=1, E=0, A=1, PD=1, N=0) \\ &= P(B=1)P(E=0)P(A=1|B=1, E=0)P(PD=1|A=1)P(N=0|A=1) \\ &= 0.001 \cdot (1 - 0.002) \cdot 0.94 \cdot 0.9 \cdot (1 - 0.75) \end{aligned}$$



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## The Alarm Network: Marginal Query

- **Question:** What is the probability that there was a break-in, but no earthquake, the police call, but your neighbor does not call?

$$\begin{aligned} P(B=1, E=0, PD=1, N=0) \\ &= \sum_{a=0}^1 P(B=1, E=0, A=a, PD=1, N=0) \\ &= P(B=1)P(E=0)P(A=1|B=1, E=0)P(PD=1|A=1)P(N=0|A=1) \\ &\quad + P(B=1)P(E=0)P(A=0|B=1, E=0)P(PD=1|A=0)P(N=0|A=0) \\ &= 0.001 \cdot (1 - 0.002) \cdot 0.94 \cdot 0.9 \cdot (1 - 0.75) \\ &\quad + 0.001 \cdot (1 - 0.002) \cdot (1 - 0.94) \cdot 0.005 \cdot (1 - 0.1) \end{aligned}$$

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## The Alarm Network: Conditional Query

- **Question:** What is the probability that the alarm went off given that there was a break-in, but no earthquake, the police call, but your neighbor does not call?

$$\begin{aligned} P(A=1|B=1, E=0, PD=1, N=0) \\ &= \frac{P(B=1, E=0, A=1, PD=1, N=0)}{\sum_{a=0}^1 P(B=1, E=0, A=a, PD=1, N=0)} \\ &= \frac{P(B=1)P(E=0)P(A=1|B=1, E=0)P(PD=1|A=1)P(N=0|A=1)}{\sum_{a=0}^1 P(B=1)P(E=0)P(A=a|B=1, E=0)P(PD=1|A=a)P(N=0|A=a)} \end{aligned}$$

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## The Alarm Network: More Queries

- What is the probability that there is a break-in given that there is an earthquake?
- What is the probability that your neighbor calls given that the alarm goes off and there is an earthquake?
- What is the probability that the police call given that the alarm goes off and your neighbor calls?
- What is the probability of a break-in given that the alarm goes off and the police call?
- What is the probability that your neighbor calls given that there is an earthquake?
- What is the probability that there is a break-in given that there is an earthquake and the alarm goes off?
- What is the probability that your neighbor calls given that the police call?

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