CS 312: Algorithms

Flow Applications

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First Application of Network Flows: Bipartite Matching

- Given a bipartite graph $G = (L \cup R, E)$, a subset of edges $M \subseteq E \subseteq L \times R$ is a matching if each node appears in at most one edge in M.
- The maximum matching problem is to find the matching with the most edges.
- We'll design an efficient algorithm for maximum matching in a bipartite graph.

Formulating it as a a network flow problem

- **Goal**: given matching instance $G = (L \cup R, E)$, create a flow network G', find a maximum flow f in G', and use f to construct a maximum matching M in G. Exercise.
  - Add a source s and sink t
  - For each edge $(u, v) \in E$, add a directed edge from u to v with capacity 1
  - Add an edge with capacity 1 from s to each node $u \in L$
  - Add an edge with capacity 1 from each node $v \in R$ to t.
  - Run F-F to get an integral max-flow f
  - Set M to the set of edges from L to R with flow $f(e) = 1$
- Claim: The set M is a maximum matching.

Correctness

There is a bijection between integral flows f of value k and matchings M of size k

1. Integral flow f of value $k \Rightarrow$ matching M of size k

- Suppose f is a flow of value k
- Let M = edges from L to R carrying one unit of flow
- There are k such edges, because the net flow across cut between L and R is k, and there are no edges from R to L
- There is at most 1 unit of flow entering $u \in L$, and therefore at most 1 unit of flow leaving u
- Since all flow values are 0 or 1, this means M has at most one edge incident to u.
- A similar argument for $v \in L$ means that M has at most one edge incident to v
- Therefore, M is a matching with size k

Correctness

2. Matching M of size $k \Rightarrow$ integral flow f of value k

- Suppose M is a matching of size k
- Send one unit of flow from s to $u \in L$ if u is matched
- Send one unit of flow from $v \in R$ to t if t is matched
- Sent one unit of flow on e if e is in M
- All other edge flow values are zero
- Verify that capacity and flow conservation constraints are satisfied, and that $v(f) = k$.

For every integer flow of value k we can construct a matching M of size k and vice versa. Therefore, a maximum integer-valued flow yields a maximum matching.

Second Application of Network Flows: Image Segmentation

- Using an expensive camera and appropriate lenses, you can get a “bokeh" effect on portrait photos in which the background is blurred and the foreground is in focus.

- But using cheap cameras in phones and appropriate software you can fake this effect. . .

Formulating the problem

Problem: given set V of pixels, classify each as foreground or background. Assume you have:

- Numeric "cost" for assigning each pixel foreground/background
- Numeric penalty for assigning neighboring pixels to different classes

Sketch of approach: other slides, board work, demo.