CS 312: Algorithms

Max-Flow Min-Cut Theorem

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Last Compiled: November 19, 2018

Step 3: F-F returns a maximum flow

We will prove this by establishing a deep connection between flows and cuts in graphs: the max-flow min-cut theorem.

- An s-t cut (A, B) is a partition of the nodes into sets A and B where $s \in A$, $t \in B$
- Capacity of cut (A, B) equals

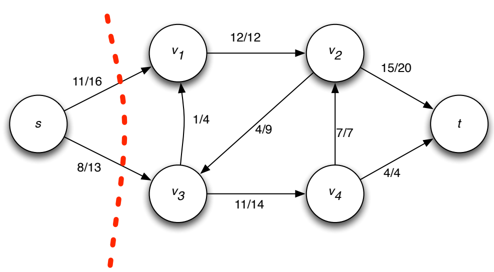
$$c(A, B) = \sum_{e \text{ from } A \text{ to } B} c(e)$$

- Flow across a cut (A, B) equals

$$f(A, B) = \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e)$$

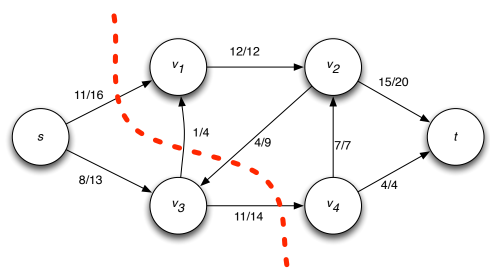
NOTE: capacity across cut is only from A to B, flow across cut *subtracts* the flow that "cycles back"

Example of Cut



Exercise: write capacity of cut and flow across cut. Capacity is 29 and flow across cut is 19.

Another Example of Cut



Exercise: write capacity of cut and flow across cut. Capacity is 34 and flow across cut is 19.

Flow Value Lemma

First relationship between cuts and flows

Lemma: let f be any flow and (A, B) be any s-t cut. Then

$$v(f) = \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e)$$

Proof omitted. Basic idea is to use conservation of flow: all the flow out of s must leave A eventually.

Corollary: Cuts and Flows

Really important corollary of flow-value lemma

Corollary: Let f be **any** s-t flow and let (A, B) be **any** s-t cut. Then $v(f) \leq c(A, B)$.

Proof:

$$\begin{aligned} v(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e) \\ &\leq \sum_{e \text{ out of } A} f(e) \\ &\leq \sum_{e \text{ out of } A} c(e) \\ &= c(A, B) \end{aligned}$$

Duality

Illustration on board

Claim If there is a flow f^* and cut (A^*, B^*) such that $v(f^*) = c(A^*, B^*)$, then

- f^* is a maximum flow
- (A^*, B^*) is a minimum cut

F-F returns a maximum flow

Theorem: The s-t flow f returned by F-F is a maximum flow.

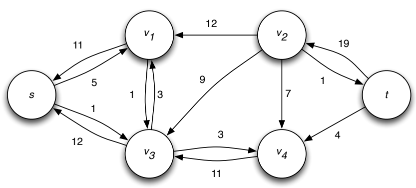
- Since f is the final flow there are no residual paths in G_f.
- Let (A, B) be the s-t cut where A consists of all nodes reachable from s in the residual graph.
  - Any edge out of A must have $f(e) = c(e)$ otherwise there would be more nodes than just A that reachable from s.
  - Any edge into A must have $f(e) = 0$ otherwise there would be more nodes than just A that reachable from s.
- Therefore

$$v(f) = \sum_{e \text{ out of} A} f(e) - \sum_{e \text{ into} A} f(e)$$
$$= \sum_{e \text{ out of} A} c(e) = c(A, B)$$

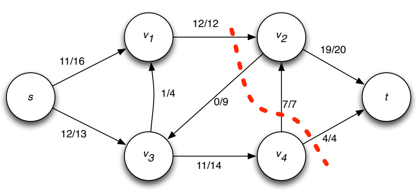
F-F finds a minimum cut

Theorem: The cut (A, B) where A is the set of all nodes reachable from s in the residual graph is a minimum-cut.

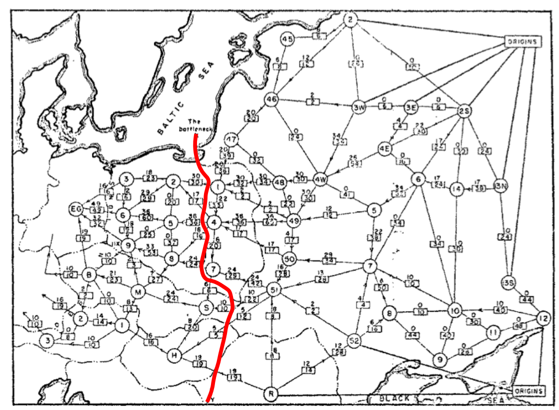
G



G_f



F-F finds a minimum cut



Capacity 163,000 tons per day [Harris and Ross 1955]