CS 312: Algorithms

Greedy: Exchange Arguments—Scheduling to Minimize Lateness

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Algorithm Design—Greedy

Greedy: make a single "greedy" choice at a time, don't look back.

	Greedy
Formulate problem	?
Design algorithm	easy
Prove correctness	hard
Analyze running time	easy

Focus is on proof techniques

- Last time: "greedy stays ahead" (inductive proof)
- This time: exchange argument

Scheduling to Minimize Lateness

- You have a very busy month: n assignments are due, with different deadlines

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Assignments:
    1: |---|            (len=1, due=2)
    2: |---|---|        (len=2, due=5)
    3: |---|---|---|    (len=3, due=6)
    4: |---|---|        (len=2, due=7)

Deadlines:
            d1          d2  d3  d4
    |---|---|---|---|---|---|---|---|---|
    0   1   2   3   4   5   6   7   8   9
```

- How should you schedule your time to "minimize lateness"?

Scheduling to Minimize Lateness

Let's formalize the problem. The input is:

- t_j = length (in days) to complete assignment j (or "job" j)
- d_j = deadline for assignment j

What does a schedule look like?

- s_j = start time for assignment j (selected by algorithm)
- $f_j = s_j + t_j$ finish time

How to evaluate a schedule?

- Lateness of assignment j is $\ell_j = \begin{cases} 0 & \text{if } f_j \leq d_j \\ f_j - d_j & \text{if } f_j > d_j \end{cases}$
- Maximum lateness $L = \max_j \ell_j$

Goal: find a schedule to make maximum lateness as small as possible

Possible Greedy Approaches

- **Note**: it never hurts to schedule assignments consecutively with no "idle time" $\Rightarrow$ schedule determined by order of assignments
- What order should we choose?
  - *Shortest Length*: ascending order of t_j.
  - *Earliest Deadline*: ascending order of d_j.
  - *Smallest Slack*: ascending order of $d_j - t_j$.
- Only *earliest deadline first* is optimal in all examples. Let's prove it is always optimal.

Exchange Argument (False Start)

Assume jobs ordered by deadline $d_1 \leq d_2 \leq \ldots \leq d_n$, so the greedy ordering is simply $A = 1, 2, \ldots, n$.

Claim: A is optimal

Proof attempt: Suppose for contradiction that A is not optimal. Then, there is an optimal solution O with $O \neq A$

- Since $O \neq A$, some pairs of jobs must be out of order (e.g. $A = 12345, O = 13254$)
- Suppose we could show this:
  - Pick two jobs in O that are out of order and swap them to match A. Call the new schedule O'. (e.g. $O = 13254 \rightarrow O' = 12354$).
  - This swap makes O' *strictly better* than O.
  - Therefore O is not optimal. Contradiction. Conclude that our assumption was wrong: A is actually optimal.

Why won't this work? O' may still be optimal. Example.

Exchange Argument (Correct)

Instead we will do this:

Suppose O optimal and $O \neq A$. Then we can modify O to get a new solution O' that is:

1. No worse than O
2. Closer to A is some measurable way

$$O(\text{optimal}) \rightarrow O'(\text{optimal}) \rightarrow O''(\text{optimal}) \rightarrow \ldots \rightarrow A(\text{optimal})$$

High-level idea: gradually transform O into A without hurting solution, thus preserving optimality.
Concretely: show 1 and 2 above.

Exchange Argument for Scheduling to Minimize Lateness

Recall $A = 1, 2, \ldots, n$. For $S \neq A$, say there is an inversion if i comes before j but $j < i$. **Claim**: if S has an inversion, S has a consecutive inversion—one where i comes immediately before j.

Main result: let $O \neq A$ be an optimal schedule. Then O has a consecutive inversion i, j. We can swap i and j to get a new schedule O' such that:

1. Maximum lateness of O' is no bigger than maximum lateness of O
2. O' has one less inversion than O

Proof:

1. On board / next slide
2. Obvious

Proof of 1

Swapping a consecutive inversion (i precedes j; $d_j \leq d_i$)

```
                |-------i-------|---j---|   O

    dj   di    |---j---|-------i-------|   O'
-----|-----|--------------------------------------->
```

Consider the lateness ℓ'_k of each job k in O':

- If $k \notin \{i, j\}$, then lateness is unchanged: $\ell'_k = \ell_k$
- Job j finishes earlier in O' than O: $\ell'_j \leq \ell_j$
- Finish time of i in O' = finish time of j in O. Therefore

$$\ell'_i = f'_i - d_j = f_j - d_i \leq f_j - d_j = \ell_j$$

Conclusion: $\max_k \ell'_k \leq \max_k \ell_k$. Therefore O' is still optimal.

Wrap-Up

For any optimal $O \neq A$ we showed that we showed that we could transform O to O' such that:

1. O' is still optimal
2. O' has one less inversion than A

$$O(\text{optimal}) \rightarrow O'(\text{optimal}) \rightarrow O''(\text{optimal}) \rightarrow \ldots \rightarrow A(\text{optimal})$$

Since there are at most $\binom{n}{2}$ inversions, by repeating the process a finite number of times we see that A is optimal.