CMPSCI 311: Introduction to Algorithms

Reductions and Intractability

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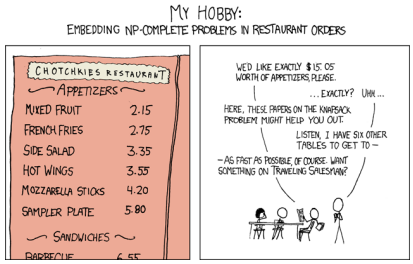
Numerical problems

Subset Sum decision problem: given n items with weights $w_1, \ldots, w_n$, is there a subset of items whose weight is exactly W?



Dynamic programming: $O(nW)$ *pseudo-polynomial* time algorithm (not polynomial in input length $n \log W$)

Subset Sum

Theorem. Subset sum is NP-complete.
Reduction from 3-SAT. (n variables, m clauses, base 10).

- All weights have $n + m$ digits
- For variable x_i, create two items t_i, f_i
  - Both have ith digit equal to 1
  - All other items have zero in this digit
  - ith digit of $W = 0 \Rightarrow$ select exactly one of t_i, f_i
- The $n + j$th digit corresponds to clause C_j
  - If $x_i \in C_j$, set $n + j$th digit of $t_i = 1$
  - If $\neg x_i \in C_j$, set $n + j$th digit of $f_i = 1$
  - Everything else 0.

- Set $n + j$th digit of $W = 3$
  - Consider a subset of items corresponding to a truth assignment (exactly one of t_i, f_i)
  - If C_j is not satisfied, then total in position $n + j$ is 0, otherwise it is 1, 2, or 3
  - Create two "dummy" items y_j, z_j with 1 in position $n + j$
  - Can select dummies to yield total of 3 in position $n + j$ iff C_j is satisfied

Subset Sum Example

Example.

$$(x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3)$$

Item	1	2	3	4	5	6
t_1	1	0	0	1	0	0
f_1	1	0	0	0	1	1
t_2	0	1	0	0	1	0
f_2	0	1	0	1	0	1
t_3	0	0	1	1	0	1
f_3	0	0	1	0	1	0
W	1	1	1	3	3	3

Item	1	2	3	4	5	6
y_1	0	0	0	1	0	0
z_1	0	0	0	1	0	0
y_2	0	0	0	0	1	0
z_2	0	0	0	0	1	0
y_3	0	0	0	0	0	1
z_3	0	0	0	0	0	1

Subset Sum Proof

- All numbers (including W) are polynomially long.
- If Φ satisfiable,
  - Select t_i if $x_i = 1$ in satisfying assigbment else select f_i.
  - Take y_j, z_j as needed.
- If subset exists with sum W
  - Either t_i or f_i is chosen. Assign x_i accordingly.
  - For each clause, at least one term must be selected, otherwise clause digit is < 3.

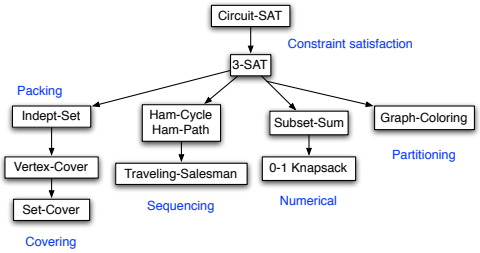
Warning

Theorem. SUBSETSUM is NP-Complete.
But Subset Sum can be tricky!

- If reducing SUBSETSUM $\leq_P X$, reduction needs to be polynomial in $\log(W)$ (number of digits).

NP-Completeness Recap

Types of hard problems:



. . . any many others. See book or other sources for more examples. You can use *any known NP-complete* problem to prove a new problem is NP-complete.