COMPSCI 311: Introduction to Algorithms

Lecture 19: Network Flow

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Review: Ford-Fulkerson Algorithm

$\triangleright$ Augment flow as long as it is possible
while there exists an s-t path P in redisual graph G_f **do**
    $f =$ Augment(f, P)
    update G_f
return f

Pearson Demo

Correctness: relate maximum flow to minimum cut

Step 3: F-F returns a maximum flow

We will prove this by establishing a deep connection between flows and cuts in graphs: the max-flow min-cut theorem.

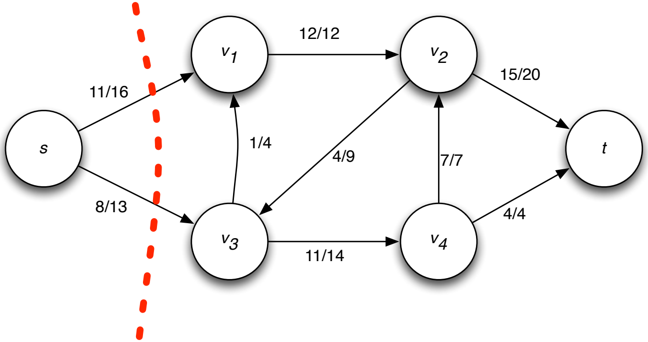
- An s-t cut (A, B) is a partition of the nodes into sets A and B where $s \in A$, $t \in B$
- Capacity of cut (A, B) equals

$$c(A, B) = \sum_{e \text{ from } A \text{ to } B} c(e)$$

- Flow across a cut (A, B) equals

$$f(A, B) = \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e)$$

Example of Cut



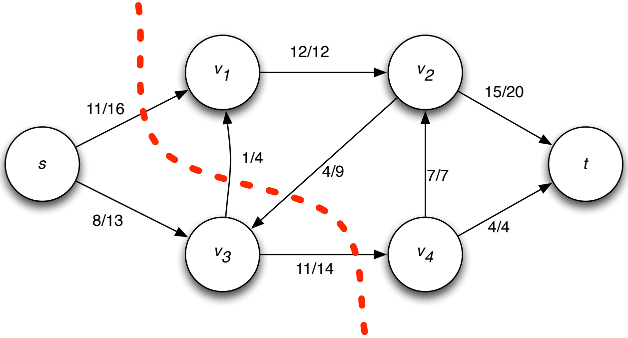
Exercise: write capacity of cut and flow across cut.

Capacity is 29 and flow across cut is 19.

Clicker Question

What is the capacity of the cut and the flow across the cut?

	Capacity	Flow
A.	16+4+9+14	11+1+3+11
B.	16+4 -9+14	11+1 -4+11
C.	16+4+14	11+1 -4+11
D.	16+4+14	11+1+11



Flow Value Lemma

First relationship between cuts and flows

Lemma: let f be any flow and (A, B) be any s-t cut. Then

$$v(f) = \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e)$$

Proof: see book. Basic idea is to use conservation of flow: all the flow out of s must leave A eventually.

Corollary: Cuts and Flows

Really important corollary of flow-value lemma

Corollary: Let f be **any** s-t flow and let (A, B) be **any** s-t cut. Then $v(f) \leq c(A, B)$.

Proof:

$$\begin{aligned} v(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e) \\ &\leq \sum_{e \text{ out of } A} f(e) \\ &\leq \sum_{e \text{ out of } A} c(e) \\ &= c(A, B) \end{aligned}$$

Duality

Illustration on board

Claim If there is a flow f^* and cut (A^*, B^*) such that $v(f^*) = c(A^*, B^*)$, then

- f^* is a maximum flow
- (A^*, B^*) is a minimum cut

Clicker

Suppose f is a flow, and there is a path from s to u in G_f, but no path from s to v in G_f. Then

A. There is no edge from u to v in G.

B. If there is an edge from u to v in G then f does not send any flow on this edge.

C. If there is an edge from u to v in G then f fully saturates it with flow.

D. None of the above.

Clicker

Suppose f is a flow, and there is a path from s to u in G_f, but no path from s to v in G_f. Then

A. There is no edge from v to u in G.

B. If there is an edge from v to u in G then f does not send any flow on this edge.

C. If there is an edge from v to u in G then f fully saturates it with flow.

D. None of the above.

F-F returns a maximum flow

Theorem: The s-t flow f returned by F-F is a maximum flow.

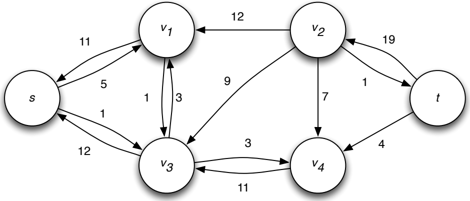
- Since f is the final flow there are no residual paths in G_f.
- Let (A, B) be the s-t cut where A consists of all nodes reachable from s in the residual graph.
  - Any edge out of A must have $f(e) = c(e)$ otherwise there would be more nodes than just A that reachable from s.
  - Any edge into A must have $f(e) = 0$ otherwise there would be more nodes than just A that reachable from s.
- Therefore

$$\begin{aligned} v(f) &= \sum_{e \text{ out of} A} f(e) - \sum_{e \text{ into} A} f(e) \\ &= \sum_{e \text{ out of} A} c(e) = c(A, B) \end{aligned}$$

F-F finds a minimum cut

Theorem: The cut (A, B) where A is the set of all nodes reachable from s in the residual graph is a minimum-cut.

G_f



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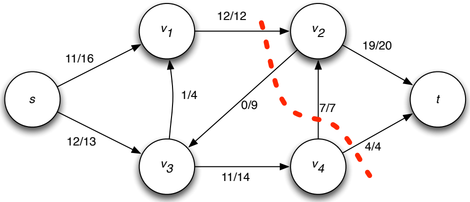
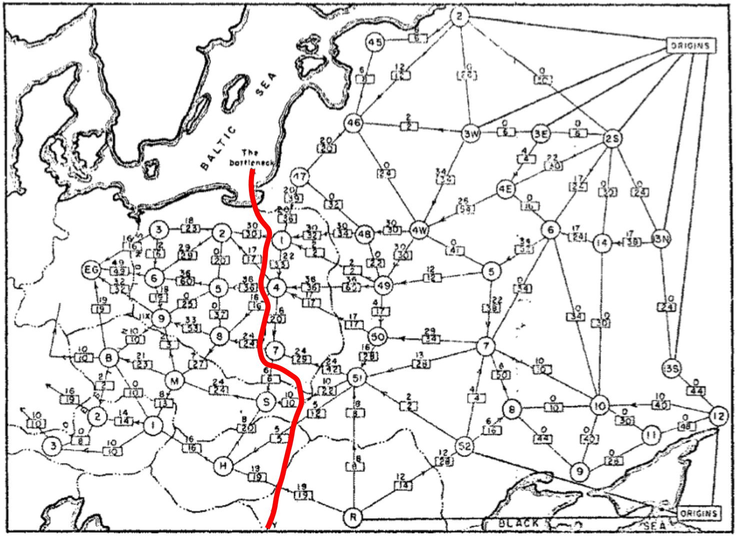


F-F finds a minimum cut



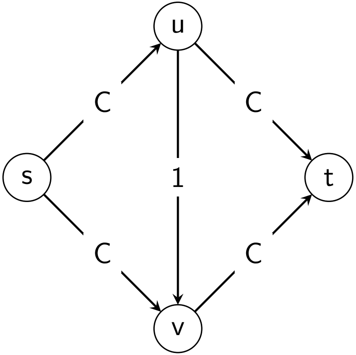
Capacity 163,000 tons per day [Harris and Ross 1955]

Ford-Fulkerson Running Time

- Flow increases at least one unit per iteration
- F-F terminates in at most C iterations, where C is the sum of capacities leaving source.
- $C \le n\,C_{\max}$, where $C_{\max}$ = maximum edge capacity
- Running time: $O(m\,n\,C_{\max})$

Is this polynomial? pseudo-polynomial (exponential in $\log C_{\max}$)

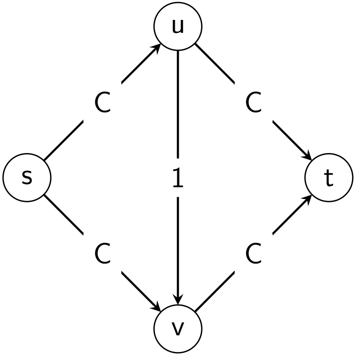
Running-Time Example



What is the smallest number of augment operations with which Ford-Fulkerson can find a maximum-flow in this graph?

A. 1
B. 2
C. 3
D. C

Improving Running Time



Good path choice will find:
$s \to u \to t$, flow C
$s \to v \to t$, flow C

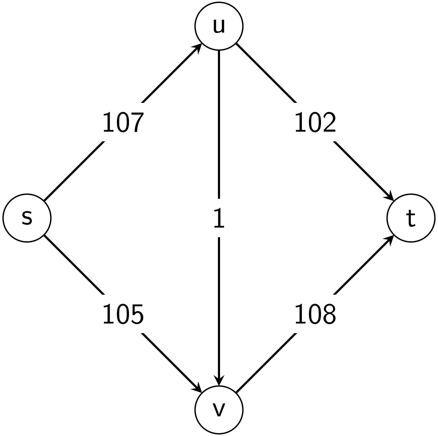
Worst-case: keep incrementing by 1:
$s \to u \to v \to t$, flow 1
$s \to v \to u \to t$, flow 1
$s \to u \to v \to t$, flow 1
...

Solution: choose good augmenting paths:

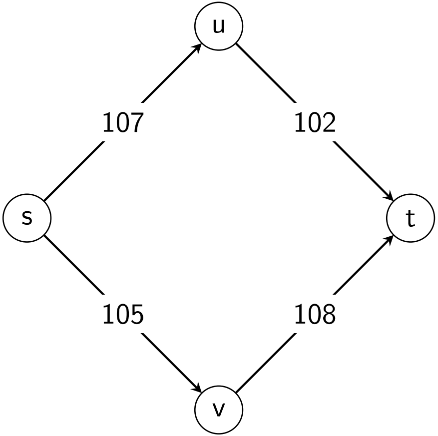
- Large bottleneck capacity: capacity-scaling algorithm
- Few edges: Edmonds-Karp, Dinitz

Capacity-scaling algorithm

Idea: ignore edges with small capacity at first



original residual graph G_f



$G_f(\Delta)$ for $\Delta = 100$. Def: only edges with residual capacity $\geq \Delta$

Capacity-scaling algorithm

Start with large Δ, divide by two in each phase

let $f(e) = 0$ for all $e \in E$
let Δ = largest power of $2 \leq C_{\max}$
while $\Delta \geq 1$ **do**
    prune residual graph G_f to $G_f(\Delta)$
    while there is augmenting $s \rightsquigarrow t$ path P in $G_f(\Delta)$ **do**
        f = Augment(f, P)
        update $G_f(\Delta)$ $\qquad \triangleright$ only $c_e \geq \Delta$
    $\Delta = \Delta/2$ $\qquad \triangleright$ refine

Capacity-Scaling: Running Time

- How many scaling phases? $\Theta(\log C_{\max})$
- How much does the flow increase at every augmentation? $\geq \Delta$
- How many augmentations per phase? $\leq 2m$
  - Can show: at end of Δ phase, flow value is within $m\Delta$ of max.
    $\implies$ at most $2m$ iterations in next phase (with $\Delta/2$)
  - (Sketch) Construct cut (A, B) as in max-flow / min-cut theorem.
  - Edges from A to B are within Δ of being saturated.
  - Edges from B to A carry less than Δ flow.
  - $\implies$ Cut capacity at most $m\Delta$ more than flow value.
- Recall: time to find augmenting path? $O(m)$
- Overall: $O(m^2 \log C_{\max})$, **polynomial**

Running Times

- Basic F-F: $O(mnC_{\max})$ **pseudo-polynomial**
  - polynomial in *magnitude*
- Capacity-scaling: $O(m^2 \log C_{\max})$ **polynomial**
  - polynomial in *number of bits*
- Edmonds-Karp: $O(m^2 n)$ **strongly-polynomial**
  - does not depend on values, only m, n
- Dinitz: $O(mn^2)$ even better
- Edmonds-Karp and Dinitz: choose *short* augmenting paths