COMPSCI 311: Introduction to Algorithms

Lecture 17: Dynamic Programming – Shortest Paths

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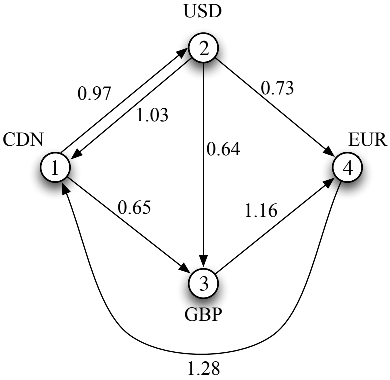
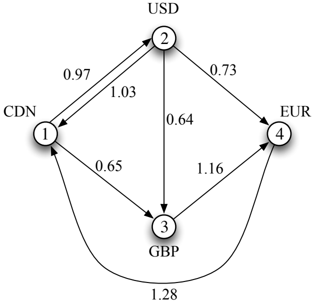
Currency Trading



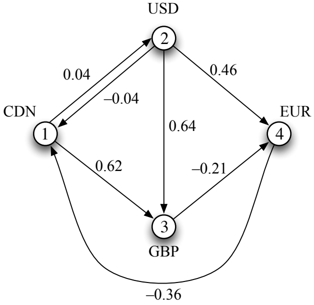
- **Problem**: given directed graph with exchange rate r_e on edge e, find $s \to t$ path P to maximize overall exchange rate $\prod_{e \in P} r_e$
- **Assumption** (no arbitrage): no cycles C such that $\prod_{e \in C} r_e > 1$.

From Rates to Costs

- Similar, but not the same as finding a shortest path.
- Let's change from rates to costs by transforming the problem.
- Let $c_e = -\log r_e$ be the *cost* of edge e



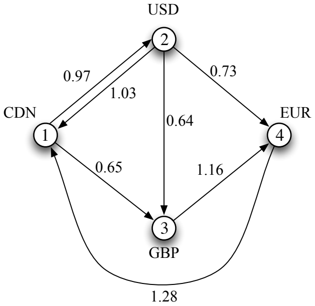
Rates

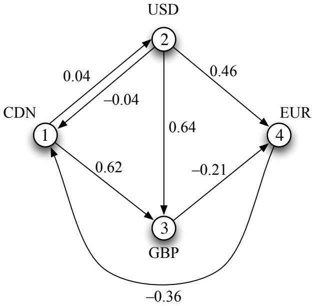


Costs

From Rates to Costs

- The cost (length) of a path becomes the negative log of its rate

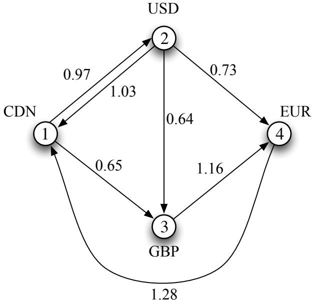


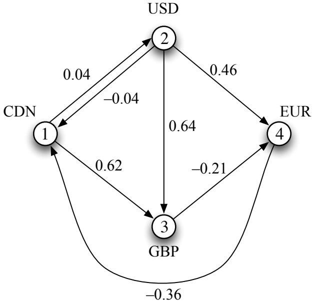


$$c_{12} + c_{23} + c_{34} = -\log r_{12} + -\log r_{23} + -\log r_{34} = -\log(r_{12} r_{23} r_{34})$$

From Rates to Costs

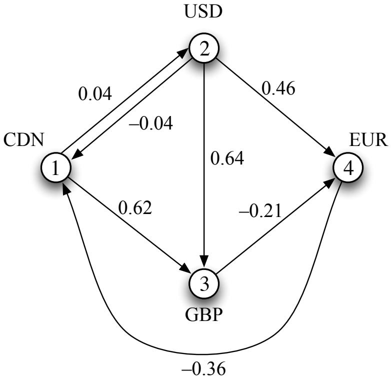
- Because $\log$ is monotone we have: lower cost $\iff$ higher rate





- **New problem**: find the $s \to t$ path of minimum cost

Currency Trading as Shortest Path Problem



- Negative edge weights!
- **Problem**: given a graph with edge weights that may be negative, find shortest $s \to t$ path
- **Assumption**: no cycle C such that $\sum_{e \in C} c_e < 0$. Why?

Dynamic Programming Approach (False Start)

- Let $\mathrm{OPT}(v)$ be the cost of the shortest $v \to t$ path
- What goes wrong with this?
- The recurrence is not well-defined, e.g., there are nodes i and j where $\mathrm{OPT}(i)$ depends on $\mathrm{OPT}(j)$ and vice versa.
- **Idea**: We can fix this by "adding a variable" to the recurrence that is always decreasing.

Bellman-Ford Algorithm

Let $\mathrm{OPT}(i, v)$ be cost of shortest $v \rightsquigarrow t$ path P **with at most i edges**

- If P uses at most $i-1$ edges then $\mathrm{OPT}(i, v) = \mathrm{OPT}(i-1, v)$
- Else $P = v \to w \rightsquigarrow t$ where $w \rightsquigarrow t$ path uses $i-1$ edges, so

$$\mathrm{OPT}(i, v) = c_{v,w} + \mathrm{OPT}(i-1, w)$$

This gives the recurrence

$$\begin{aligned}
\mathrm{OPT}(i, v) &= \min\left\{\mathrm{OPT}(i-1, v),\ \ \min_{w \in V}\{c_{v,w} + \mathrm{OPT}(i-1, w)\}\right\} \\
\mathrm{OPT}(0, t) &= 0 \\
\mathrm{OPT}(0, v) &= \infty \text{ if } v \neq t
\end{aligned}$$

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With negative edge lengths, paths can get *shorter* as we include more edges.

Assuming all cycles have positive cost and $m > n$, what is the largest possible number of edges in a shortest-length path from v to t?

A. n

B. m

C. $n - 1$

D. $m - 1$

Bellman-Ford

$$\mathrm{OPT}(i, v) = \min \left\{ \mathrm{OPT}(i-1, v), \;\; \min_{w \in V} \{ c_{v,w} + \mathrm{OPT}(i-1, w) \} \right\}$$

Subproblems? $\mathrm{OPT}(i, v)$ for $i = 1$ to $n - 1$, $v \in V$
(**Fact**: shortest path has at most $n - 1$ edges)

Shortest-Path(G, s, t)

n = number of nodes in G
Create array M of size $n \times n$
Set $M[0, t] = 0$ and $M[0, v] = \infty$ for all other v
for $i = 1$ to $n - 1$ **do**
  for all nodes v in any order **do**
    Compute $M[i, v]$ using the recurrence above

Running time? $O(n^3)$. Better analysis $O(mn)$. Example

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Suppose there is some iteration i for which $M[i, v] = M[i-1, v]$ for all v. Then

A. There is a negative cycle in the graph.

B. We can terminate the algorithm after the ith iteration, because no future values will change.

C. There are no negative edge costs in the graph.

D. The graph is undirected.

Bellman-Ford-Moore: Efficient Implementation

- Store only one column: M array $\to$ d vector
- Only consider neighbors w whose value changed
- Keep track of shortest path using successor array

Shortest-Path(G, t)

set $d[t] = 0$ and $d[v] = \infty$ for all $v \neq t$
set succ$[v] =$ `null` for all v
for $i = 1$ to $n - 1$ **do**
  for all nodes $w \neq t$ **do**
    if w updated in last iteration **then**
      for all $(v, w) \in E$ **do**
        if $c_{v,w} + d[w] < d[v]$ **then**
          $d[v] = c_{v,w} + d[w]$
          succ$[v] = w$

- Space? $O(m + n)$, time $O(mn)$

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Suppose we remove the assumption that there are no negative cycles, and find that $\mathrm{OPT}(n, v) < \mathrm{OPT}(n-1, v)$ for some node v. Then

A. There is a negative cycle on some $v \rightsquigarrow t$ path in the graph.

B. There are no negative edge costs in the graph.

C. There is a negative cycle on some $t \rightsquigarrow v$ path in the graph.

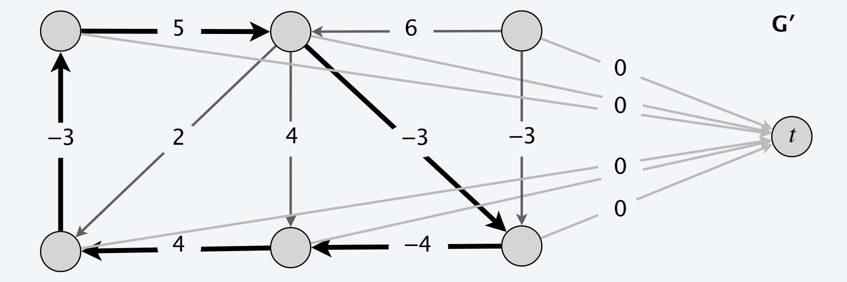
D. There are no negative cycles in the graph.

Negative Cycles

- How to detect negative-weight cycles?
  - Suppose $\mathrm{OPT}(n, v) < \mathrm{OPT}(n-1, v)$. Then there is a negative cycle on some $v \rightsquigarrow t$ path, since shortest paths have at most $n-1$ edges in the absence of negative cycles.
  - Suppose $\mathrm{OPT}(n, v) = \mathrm{OPT}(n-1, v)$ for all v. Then the algorithm will not update after the nth iteration
    $\implies$ $\mathrm{OPT}(n+i, v) = \mathrm{OPT}(n-1, v)$ for all $i \geq 0$
    $\implies$ no negative cycles on any $v \rightsquigarrow t$ path.
- **Fact**: there is a negative cycle on some $v \rightsquigarrow t$ path iff $\mathrm{OPT}(n, v) < \mathrm{OPT}(n-1, v)$ for some v.
- Detect negative cycles by running for one more iteration to see if some value decreases!

Detecting Negative-Weight Cycles

But this only detects cycles on paths to a fixed target node t. How to find a negative-weight cycle anywhere in the graph?



Add a dummy target node.