CMPSCI 311: Introduction to Algorithms
Second Midterm Exam

April 13, 2017.

Name: ____________________________________ ID: ________________________

Instructions:

- Answer the questions directly on the exam pages.
- Show all your work for each question. Providing more detail including comments and explanations can help with assignment of partial credit.
- If you need extra space, use the blank pages at the end of the exam.
- No books, notes, calculators or other electronic devices are allowed. Any cheating will result in a grade of 0.
- If you have questions during the exam, raise your hand.

Question 1. (*10 points*) Indicate whether each of the following statements is TRUE or FALSE. No justification required.

1.1 (*2 points*): *Given a flow network where all the edge capacities are even integers, the algorithm will require at most $C/2$ iterations, where C is the total capacity leaving the source s.*

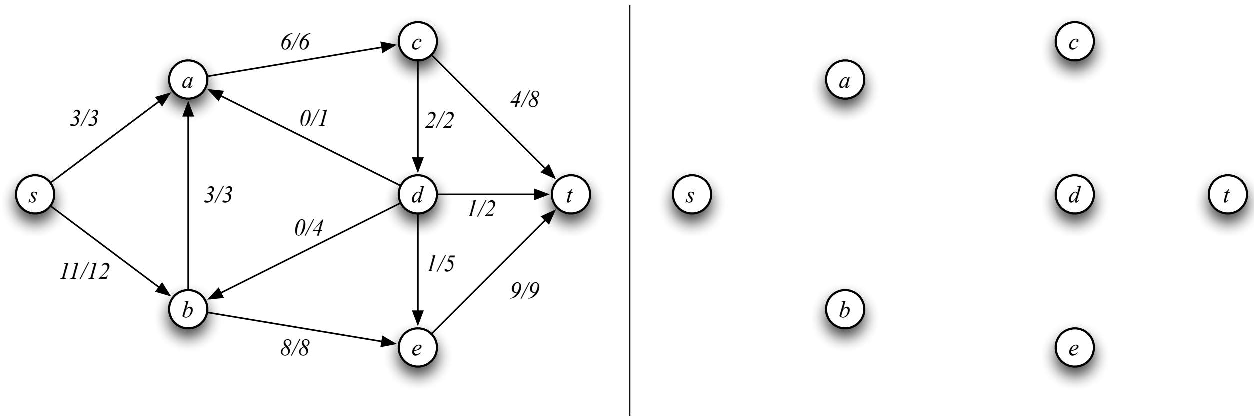
1.2 (*2 points*): *Suppose that $T(n) = 2T(n-1) + O(1)$ and $T(1) = 1$. Then $T(n) = O(n^2)$.*

1.3 (*2 points*): *You analyze the recursion tree for an algorithm with recurrence $T(n) \leq aT(n/b) + cn$ and find that amount of work is decreasing at each level of the recursion tree. The running time is $O(n)$.*

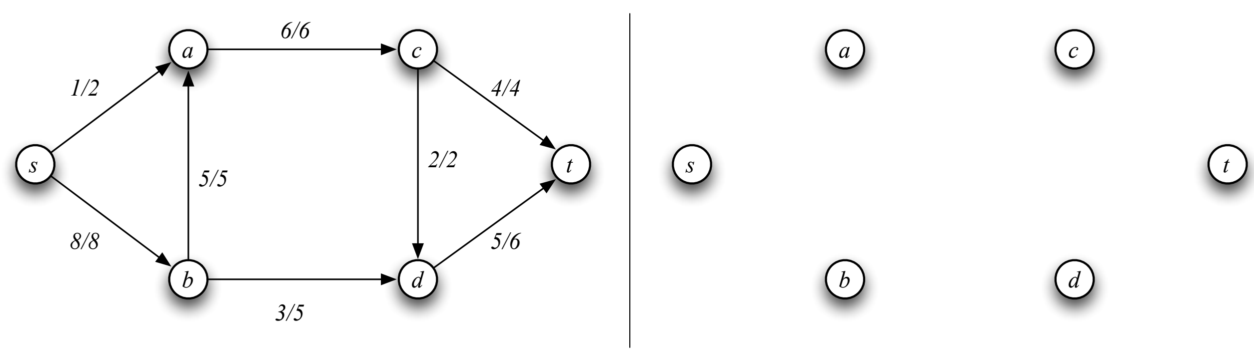
1.4 (*2 points*): *Suppose f is a flow of value 100 from s to t in a flow network G. The capacity of the minimum s-t cut in G is equal to 100.*

1.5 (*2 points*): *The recurrence $\mathrm{OPT}(j) = \max_{1 \leq i \leq j} \{A[i] - B[j-i] + \mathrm{OPT}(j-i)\}$, where A and B are fixed input arrays, will lead to an $O(n)$ dynamic programming algorithm.*

Question 2. (*10 points*) Each of the two graphs below displays a flow f in a graph G. Each edge e is labeled with two values $f(e)/c(e)$ (i.e., flow / capacity). You may use the right panel in each figure to draw the *residual graph* G_f to help answer the questions.



1. What is the value of the flow?
2. What is the residual capacity of the edge (e, d) in G_f?
3. Indicate a minimum cut and its capacity.
4. Is there an $s \to t$ path in the residual graph? If so, indicate one.
5. Is the flow a maximum flow?



1. What is the value of the flow?
2. What is the residual capacity of edge (b, d) in G_f?
3. List a minimum cut and its capacity.
4. Is there an $s \to t$ path in the residual graph? If so, indicate one.
5. Is the flow a maximum flow?

Question 3. (*10 points*) You are working for the UMass fake news agency, and plan to write a story about the massive growth in crowd size at commencement over the history of UMass. In reality, crowd size fluctuates from year to year: there are n years of records, and the crowd size in year i is $s(i)$. Your plan is to find the two years i and j such that $i < j$ and the difference $s(j) - s(i)$ is as large as possible, and print photographs of the crowds in those two years to demonstrate the crowd growth.

3.1 (*1 points*): *Suppose* $n = 4$ *and* $s(1) = 3$, $s(2) = 1$, $s(3) = 4$, $s(4) = 5$ *What years will you select in this example?*

$i =$ $j =$

Consider the following outline for an algorithm to solve the problem. Assume that n is a power of 2, and, for simplicity, assume your algorithm returns only the maximum *value* of $s(j) - s(i)$, and not the years themselves. Let $L = \{1, \ldots, n/2\}$ and $R = \{n/2 + 1, \ldots, n\}$.

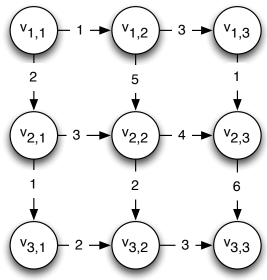
1. Recursively find the biggest increase $s(j) - s(i)$ for $i, j \in L$, and call this value δ_L
2. Recursively find the biggest increase $s(j) - s(i)$ for $i, j \in R$, and call this value δ_R
3. ???

3.2 (*2 points*): *Assuming Step 3 takes linear time, write a recurrence for the running time* $T(n)$ *of this algorithm.*

3.3 (*2 points*): *Give the running time of the algorithm (i.e., state the solution to your recurrence).*

3.4 (*5 points*): *Describe how to complete Step 3 of the algorithm using* $O(n)$ *time steps.*

Question 4. (*10 points*) In this question, we develop an alternative shortest path algorithm for a specific type of graph. In a *grid graph* there are $n = k^2$ nodes labelled $v_{i,j}$ where i and j range between 1 and k. For each $i < k$ there is a "down" edge of length $d_{i,j}$ from $v_{i,j}$ to $v_{i+1,j}$. For each $j < k$ there is an "across" edge of length $a_{i,j}$ from $v_{i,j}$ to $v_{i,j+1}$. For example, a possible graph for $n = 9$ is



and in this graph, for example, the across edge from $v_{2,1}$ to $v_{2,2}$ has length $a_{2,1} = 3$.

4.1 (*2 points*): *What is the length of the shortest path from $v_{1,1}$ to $v_{3,3}$ in the above example?*

4.2 (*4 points*): *Let $OPT(i,j)$ be the length of the shortest path from $v_{i,j}$ to $v_{k,k}$. Give a formula for $OPT(i,j)$ in terms of $OPT(i+1,j)$ and $OPT(i,j+1)$ when $i,j \leq k$. Your recurrence should have four different cases:*

Case 1. $i < k, j < k \quad OPT(i,j) =$

Case 2. $i < k, j = k \quad OPT(i,j) =$

Case 3: $i = k, j < k \quad OPT(i,j) =$

Case 4: $i = k, j = k \quad OPT(i,j) =$

4.3 (*4 points*): *Describe an algorithm to compute $OPT(1,1)$ and state its running time. You do not need to prove correctness.*

Question 5. (*10 points*) Recall that an independent set in a graph is a subset of nodes S such that no two nodes in S are connected by an edge. In this problem, we seek a maximum-weight independent set in a tree $T = (V, E)$ with root node r and node weights $w(v)$ for all $v \in V$.

For any node v, let T_v denote the subtree rooted at v. Define $\text{OPT}(v)$ to be the maximum-weight independent set in T_v. For the base case, it is clear that $\text{OPT}(u) = w(u)$ if u is a leaf.

5.1 (*3 points*): *Now, let $C(v)$ be the children of v. Complete the recurrence below for a non-leaf node u:*

$$\text{OPT}(u) = \max \begin{cases} w(u) + \sum_{v \in C(u)} \sum_{x \in C(v)} \text{OPT}(x) & \text{(Case 1)}, \\ & \\ & \text{(Case 2)}. \end{cases}$$

5.2 (*2 points*): *Two different alternatives are being considered in Case 1 and Case 2. Explain precisely what they are.*

5.3 (*3 points*): *Consider a dynamic programming algorithm to compute $OPT(r)$ for the root node r. Indicate what size array is needed, and describe a valid order to fill in the array entries. (You do not need to give pseudocode).*

5.4 (*2 points*): *Let n be the number of nodes and let d be the maximum degree of any node. Give a running time bound for the algorithm in terms of n and d.*

5.5 (*2 points*): *(Extra credit) Give a tight running time bound that does not depend on d and explain why it is correct.*

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