CMPSCI 311: Introduction to Algorithms
Practice Final Exam

Name: _____ ID: _____

Instructions:

- Answer the questions directly on the exam pages.
- Show all your work for each question. Providing more detail including comments and explanations can help with assignment of partial credit.
- If the answer to a question is a number, *unless the problem says otherwise*, you may give your answer using arithmetic operations, such as addition, multiplication, "choose" notation and factorials (e.g., "$9 \times 35! + 2$" or "$0.5 \times 0.3/(0.2 \times 0.5 + 0.9 \times 0.1)$" is fine).
- If you need extra space, use the back of a page.
- No books, notes, calculators or other electronic devices are allowed. Any cheating will result in a grade of 0.
- If you have questions during the exam, raise your hand.

Question 1. (*10 points*) **True or False?** Indicate whether each of the following statements is TRUE or FALSE. No justification required.

1.1 (*2 points*): $\sum_{i=1}^{n} 2^i = \Theta(2^n)$.

1.2 (*2 points*): *A dynamic program that implements the following recursive form can be used to solve the subset sum problem, which asks to find a subset S of numbers from $x_1, \ldots, x_n$ (all non-negative) with maximum weight subject to not exceeding a given number W.*

$$OPT(i, w) = \max\{OPT(i-1, w), x_i + OPT(i-1, w - x_i)\}$$

1.3 (*2 points*): *For any flow network, and any two vertices s, t there is always a flow of at least 1 from source s to target t.*

1.4 (*2 points*): *The recurrence $T(n) = 2T(n-1) + O(1)$ solves to $\Theta(n^2)$.*

1.5 (*2 points*): *Suppose $X \leq_P Y$ and X is solvable in polynomial time. Then $P = NP$.*

Question 2. (*20 points*) **Short Answer.** Answer each of the following questions in at most two sentences.

2.1 (*4 points*): *In a weighted graph G where all edges have weight* 1*, how can we use Djikstra's algorithm to find a minimum spanning tree?*

2.2 (*4 points*): *Solve the recurrence* $T(n) = 3T(n/2) + O(n)$.

2.3 (*4 points*): *Suppose a dynamic programming algorithm creates an* $n \times m$ *table and to compute each entry of the table it takes a minimum over at most* m *(previously computed) other entries. What would the running time of this algorithm be, assuming there is no other computations.*

2.4 (*4 points*): *Suppose a connected graph G on n vertices has exactly two cycles. How many edges does it have?*

2.5 (*4 points*): *Suppose G is a graph with distinct edge weights and T is a minimum-spanning tree. Let $e \in T$. Describe how to find a cut $(S, V - S)$ such that e is the smallest weight edge crossing the cut.*

Question 3. (*12 points*) Consider the longest increasing subsequence problem defined as follows. Given a list of numbers $a_1, \ldots, a_n$ an increasing subsequence is a list of indices $i_1, \ldots, i_k \in \{1, \ldots, n\}$ such that $i_1 < i_2 < \ldots, i_k$ and $a_{i_1} \leq a_{i_2} \leq \ldots, \leq a_{i_k}$. The longest increasing subsequence is the longest list of indices with this property.

3.1 (*1 points*): *What is the longest increasing subsequence of the list* $5, 3, 4, 8, 7, 10$*?*

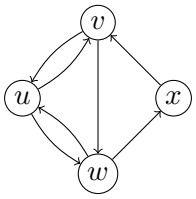
3.2 (*2 points*): *Consider the greedy algorithm that chooses the first element of the list, and then repeatedly chooses the next element that is larger. Is this a correct algorithm? Either prove its correctness or provide a counter example.*

3.3 (*2 points*): *Consider the greedy algorithm that chooses the smallest element of the list, and then repeatedly chooses the smallest element that comes after this chosen one. Is this a correct algorithm? Either prove its correctness or provide a counterexample.*

3.4 (*2 points*): *Consider a divide and conquer strategy that splits the list into the first half and second half, recursively computes* $L = (\ell_1, \ldots, \ell_{k_L}), R = (r_1, \ldots, r_{k_R})$ *the longest increasing subsequences in each half, and then, if the last chosen element in the first half is less than the first chosen index in the second half (i.e.* $a_{\ell_{k_L}} \leq a_{r_1}$*) returns* $L \cdot R$*, otherwise it returns the longer of* L *and* R*. Is this a correct algorithm? either prove its correctness of provide a counterexample.*

3.5 (*5 points*): *Design a dynamic programming algorithm for longest increasing subsequence. Prove its correctness and analyze its running time.*

Question 4. (*10 points*) In this problem we investigate the feedback vertex-set problem. Given a directed graph (which may contain cycles), a feedback vertex set S is a set of nodes such that removing the nodes S from G leaves a graph with no cycles. Another way to put this is that every cycle in G contains at least one node from S.



4.1 (*2 points*): *In the above graph, what is the size of the minimum feedback vertex set?*

4.2 (*2 points*): *True or False. A directed graph with a topological ordering always has a feedback vertex-set of size zero.*

4.3 (*6 points*): *Prove that the decision version of feedback vertex-set is NP-complete. That is given a directed graph and an integer k, decide whether the graph has a feedback vertex-set with cost at most k.* ***Hint:*** *reduce from vertex cover.*

Question 5. (*10 points*) In this problem we investigate vertex-capacitated flow networks. We are given a directed graph $G = (V, E)$ with source s and sink t and a capacity c_v for each $v \in V$. We want an $s-t$ flow f that satisfies the usual conservation of flow constraints, but instead of satisfying edge-capacity constraints, satisfies the vertex capacity constraints $f(v) \leq c_v$. Here $f(v) = \sum_{(u,v)\in E} f_{u,v}$ is the total flow entering the node v. The goal is to design an algorithm for computing a maximum $s - t$ flow in a vertex-capacitated network.

5.1 (*3 points*): *Draw a directed graph G with clearly labeled source s and sink t, where if we consider the usual edge-capacitated version of the problem (with edge capacities $c_e = 1$) we get a maximum flow with a different value than if we consider the vertex capacitated version of the problem (with vertex capacities $c_v = 1$).*

5.2 (*7 points*): *Design a polynomial time algorithm for computing the maximum flow in a node-capacitated network. Prove that the algorithm is correct and analyze its running time.*