CS 103: Lecture 8 Matching Markets

Dan Sheldon

October 16, 2015

Announcements

- HW 1 and 2 back now
- HW 3 assigned today, due next Thursday
- Midterm... stay posted
- Blog posts are great!

Plan for today

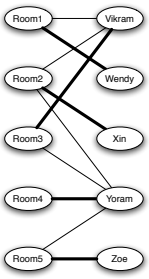
- Matching markets
  - Review
  - Market-clearing prices

Review

- Perfect matching
- Constricted set

Explain to a partner

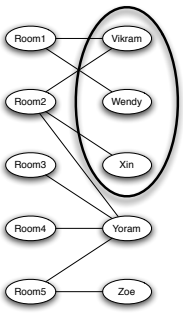
Perfect Matching



Perfect matching: each node L assigned to a single node on the R *to which it is connected by an edge*

- Students / rooms
- Baristas / shifts
- Classes / rooms
- Planes / gates

Constricted Set



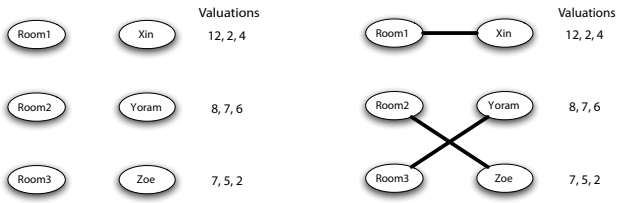
S = set of nodes on the right
$N(S)$ = set of left nodes with an edge to a node in S

S is **constricted** if $N(S)$ is smaller than S

Matching Theorem (König 1931 / Hall 1935)

Matching Theorem: if a bipartite graph has no perfect matching, then it must contain a constricted set.

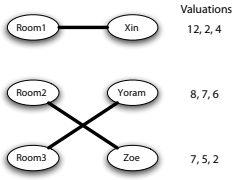
Valuations



Valuations

Value $12 + 6 + 5 = 23$

Optimal Assignment



Total valuation of an assignment: sum of each individual's valuation

Optimal assignment: assignment that maximizes total valuation

Is this assignment optimal? Yes.

Do people get most preferred rooms? No.

Prices and Payoffs

Introduce *prices* as a mechanism to resolve contention

Setup on board

- Sellers
- Prices
- Buyers
- Valuations
- Payoff

Preferred-sellers

Definitions

- Say that i is a **preferred seller** of j if buying from i maximizes j's payoff
- **Preferred-seller graph**: graph with an edge form each buyer to *all* preferred sellers

Examples on board

Market-clearing prices

Defintion: A set of prices is **market-clearing** if there is a perfect matching in the preferred seller graph (can sell all goods using preferred sellers).

Questions

- Can we always find market-clearing prices? (yes)
- What properties do they have? (optimality!)

Optimality of Market-Clearing Prices

Claim: for any set of market-clearing prices, a perfect matching in the preferred-seller graph is an *optimal* assignment of sellers to buyers (highest possible total valuation)

Proof on board

Existence of Market-Clearning Prices

Claim: for any set of buyer valuations, there is a set of market-clearing prices.

Proof by algorithm!

Algorithm to find Market-Clearing Prices

Start with all prices equal to zero, then adjust the prices in a sequence of rounds.

In each round, do the following:

- Construct preferred-seller graph
- If there is a perfect matching, **done**
- Otherwise, find a constricted set S of buyers, with neighbors $N(S)$
- Each seller in $N(S)$ raises price by one
- Reduce all prices by same amount until smallest prices is zero (if needed)

Execute the algorithm on the board

Does this always work?

Clearly, if the algorithm terminates, it produces a set of market-clearing prices?

But can it go on forever?

We will now prove that it cannot. **proof on board**