Supplementary Material: Inference in a Partially Observed Queuing Model with Applications in Ecology

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1. Proof of Log-Concavity

Proof of Theorem 2.

Proof. We can factor the pmf of δ as follows:

$$p(\delta) = a(\delta)b(\delta)\prod_{n_k \in n^+} c_k(\delta)e(\delta)\prod_{n_l \in n^-} d_l(\delta)e(\delta)^{-1}$$

$$a(\delta) = \prod_{p_i \in p^+, q_i \in q^+} \frac{p_i^{q_i+\delta}}{(q_i+\delta)!}$$

$$b(\delta) = \prod_{p_j \in p^-, q_j \in q^-} \frac{p_j^{q_j-\delta}}{(q_j-\delta)!}$$

$$c_k(\delta) = \frac{(n_k+\delta)!}{(n_k+\delta-y_k)!}$$

$$d_l(\delta) = \frac{(n_l-\delta)!}{(n_l-\delta-y_l)!}$$

$$e(\delta) = (1-\alpha)^\delta$$

Where $\{p^+, q^+, n^+\}$ and $\{p^-, q^-, n^-\}$ represent the subsets of $\mathbf{p}, \mathbf{q}$, and $\mathbf{n}$ which change positively and negatively under $\mathbf{z}$ accordingly. Since the product of log concave functions is also log concave, it is thus sufficient to demonstrate that each of the factors of $\mathcal{L}(\delta)$ is log concave. Observe that the inner part of $a(\delta)$ is the form of $e^\lambda\text{Poisson}_\lambda(k)$ where $\lambda = p_i$ and $k = q_i + \delta$. Since the Poisson is log concave and so is e^λ, $a(\delta)$ is also log concave in δ. By an identical argument, so is $b(\delta)$.

For $e(\delta)$, note that $\log(1-\alpha)^\delta = \delta\log(1-\alpha)$, which is linear in δ and therefore $e(\delta)$ is log concave in δ, as is $e(\delta)^{-1}$.

The proof of concavity for $c_k(\delta)$ and $d_l(\delta)$ is below:

$$c_k(\delta) = \frac{(n_k+\delta)!}{(n_k+\delta-y_k)!}$$

Let $n' = n_k + \delta$

$$c_k(n') = \frac{n'!}{(n'-y_k)!}$$

by construction, $n' = n_k + \delta \geq y_k$

to show $c_k(n')$ is log concave, we must show:

$$\begin{aligned}
c_k(n')^2 &\geq c_k(n'-1)c_k(n'+1) \\
\frac{c_k(n')}{c_k(n'+1)} &\geq \frac{c_k(n'-1)}{c_k(n')} \\
\frac{n'!}{(n'-y_k)!}\frac{(n'+1-y_k)!}{(n'+1)!} &\geq \frac{(n'-1)!}{(n'-1-y_k)!}\frac{(n'-y_k)!}{n'!} \\
\frac{n'+1-y_k}{n'+1} &\geq \frac{n'-y_k}{n'} \\
1-\frac{y_k}{n'+1} &\geq 1-\frac{y_k}{n'}
\end{aligned}$$

Thus $c_k(n')$ and, by extension, $c_k(\delta)$ are log concave. A similar argument shows that $d_l(\delta)$ is log concave as well. Then we have shown that $p(\delta)$ is a product of log concave functions and therefore $p(\delta)$ is also log concave. □