Data Streams & Communication Complexity

Lecture 3: Communication Complexity and Lower Bounds

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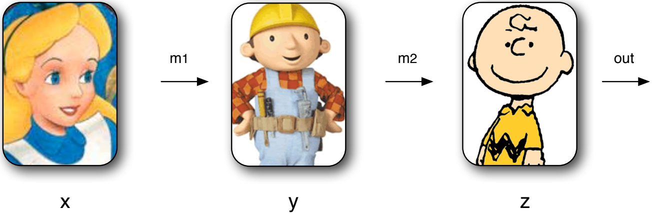

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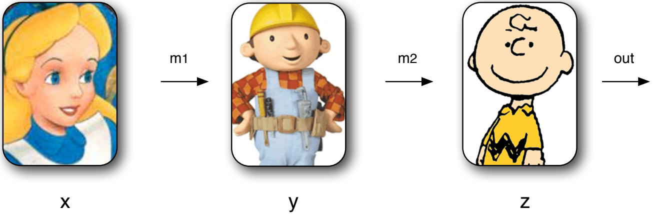

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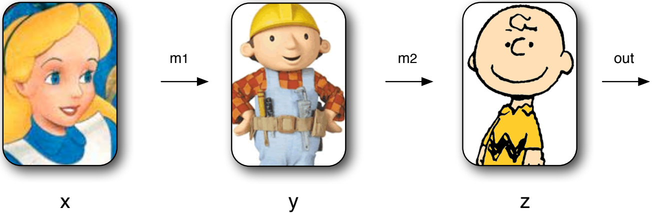

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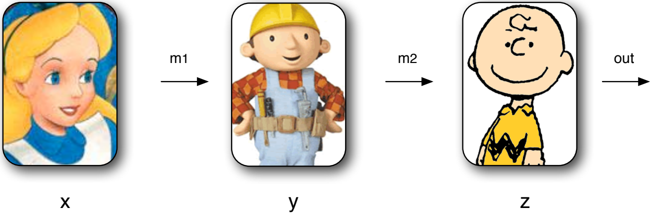

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  - *Random:* $m_1(x, r)$, $m_2(m_1, y, r)$, $\text{out}(m_2, z, r)$ where r is public random string. Require $\mathbb{P}_r[\text{out}(m_2, z, r) = P(x, y, z)] \geq 9/10$.

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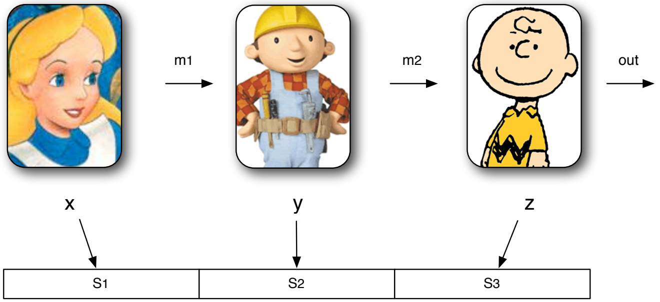
- Let Q be some stream problem. Suppose there's a reduction $x \to S_1$, $y \to S_2$, $z \to S_3$ such that knowing $Q(S_1 \circ S_2 \circ S_3)$ solves $P(x, y, z)$.



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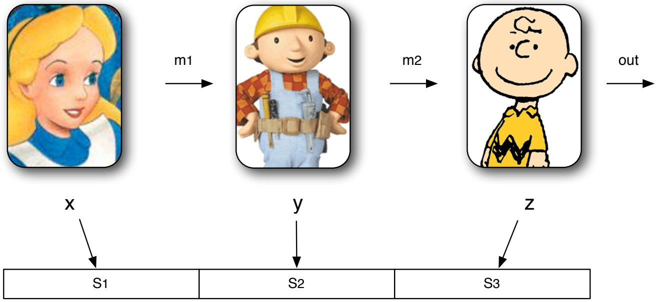

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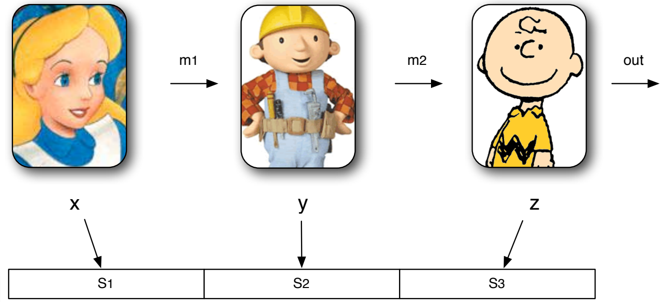

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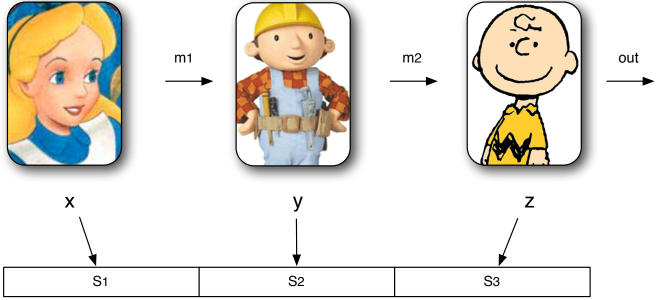

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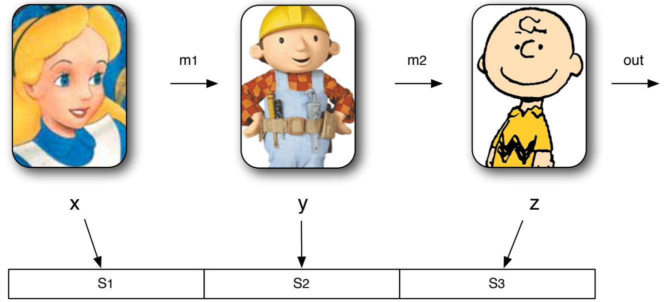

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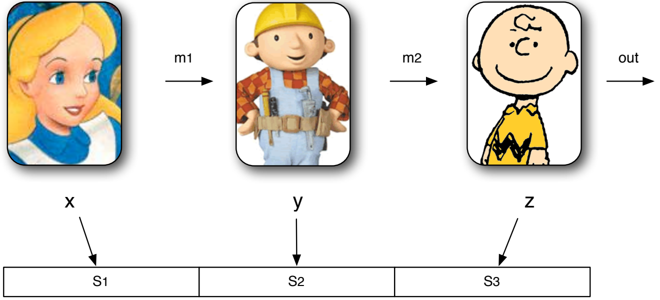

- An s-bit stream algorithm $\mathcal{A}$ for Q yields $2s$-bit protocol for P: Alice runs $\mathcal{A}$ of S_1; sends memory state to Bob; Bob instantiates $\mathcal{A}$ with state and runs it on S_2; sends state to Charlie who finishes running $\mathcal{A}$ on S_3 and infers $P(x, y, z)$ from $Q(S_1 \circ S_2 \circ S_3)$.

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- Hence, a lower bound of L on the communication required for P implies $s \geq L/(t-1)$ bits of space are required to solve Q.

Outline of Lecture

Classic Problems and Reductions

Information Statistics Approach

Hamming Approximation

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- How many bits need to be sent by Alice for Bob to determine $\textsc{Index}(x, j)$ with probability $9/10$? $\Omega(n)$

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- Then $\text{median}(S_1 \cup S_2) = 2j + x_j$ and this determines $\textsc{Index}(x,j)$.
- An s-space algorithm implies an s-bit protocol so

$$s = \Omega(n)$$

by the communication complexity of indexing.

Multi-Party Set-Disjointness

- Consider a $t \times n$ matrix where column has weight $0, 1$, or t, e.g.,

$$C = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

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$$\left(\begin{array}{cccccc} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right) \longrightarrow \{4, 1, 4, 5, 2, 4, 4\}$$

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- An s-space 2-approximation implies an $s(t-1)$ bit protocol so

$$s = \Omega(n/t^2) = \Omega(n^{1-2/k})$$

by the communication complexity of set-disjointness.

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- Consider 2 binary vectors $x, y \in \{0,1\}^n$, e.g.,

$$x = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

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$$s = \Omega(n) = \Omega(1/\epsilon^2)$$

by communication complexity of approximating Hamming distance.

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- We'll first give some definitions and then run through an example.

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- *Mutual Information:* $I(X : Y) = H(X) - H(X|Y)$

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- *Entropy:* $H(X) := \sum_i -\mathbb{P}[X=i] \lg \mathbb{P}[X=i]$
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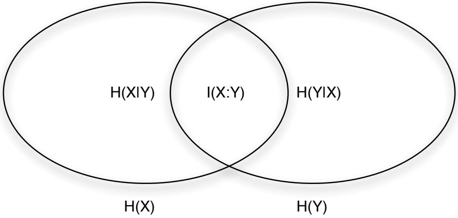

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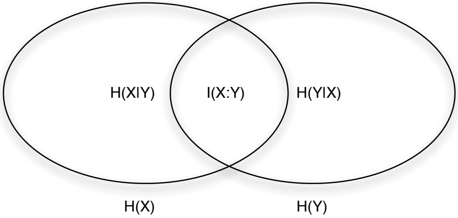

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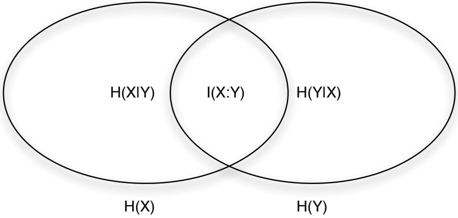

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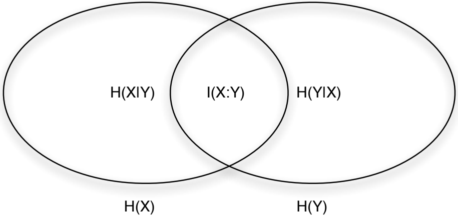

- Useful Facts:
  - If X takes at most 2^ℓ values, then $H(X) \leq \ell$.
  - If X and Y are independent, then $I(XY : Z) \geq I(X : Z) + I(Y : Z)$.

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- *Note 2:* The message in $\Pi_{\textsc{Index}}$ on input $X \in_R \{0,1\}^n$ is distributed identically to the message in $\Pi_{\textsc{Echo},i}$ on input $B \in_R \{0,1\}$.

Relating Information Cost of INDEX and ECHO

- Since $X_1, X_2, \ldots, X_n$ are independent:

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where $H_2(p) = -p \lg p - (1-p)\lg(1-p)$.

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- Hence, $\mathrm{cost}(\Pi_{\mathrm{INDEX}}) \geq (1 - H_2(\delta))n$.

Outline for $\textsc{Disj}_t$ Lower Bound

- Express $\textsc{Disj}_t$ in terms of $\textsc{And}_t$ where $\textsc{And}_t(x_1, \ldots, x_t) = \prod_i x_i$:

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- Result follows by showing $\text{icost}(\Pi_{\textsc{And}_t, i} | D) = \Omega(1/t)$.

Outline

Classic Problems and Reductions

Information Statistics Approach

Hamming Approximation

Hamming Approximation Lower Bound

Some communication results can be proved via a reduction from other communication results.

Theorem

If Alice and Bob have $x, y \in \{0,1\}^n$ and Bob wants to determine $\Delta(x, y)$ up to $\pm\sqrt{n}$ with probability $9/10$, then Alice must send $\Omega(n)$ bits.

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$$x_i = I[\text{sign}(r.z) = +] \quad \text{and} \quad y_i = I[\text{sign}(r_j) = +]$$

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$$z_j = 0 \Rightarrow \mathbb{E}[\Delta(x,y)] = n/2$$
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- Hence, a $\pm\sqrt{n}$ approx. of $\Delta(x,y)$ determines z_j with prob. $> 9/10$.

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Claim

Let A be the event $A = \{\text{sign}(r.z) = r_j\}$. For some constant $c > 0$,

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  - $\mathbb{P}[A|s=0] = 1$ since $s = 0 \Rightarrow r.z = r_j \Rightarrow A$.

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- If $z_j = 0$: $\text{sign}(r.z)$ and r_j are independent so $\mathbb{P}[A] = 1/2$.
- If $z_j = 1$: Let $s = r.z - r_j$, the sum of an even number ($\ell = t/2 - 1$) of independent ± 1 values. Then,

$$\mathbb{P}[A] = \mathbb{P}[A|s = 0]\,\mathbb{P}[s = 0] + \mathbb{P}[A|s \neq 0]\,\mathbb{P}[s \neq 0]$$

  - $\mathbb{P}[s = 0] = \binom{\ell}{\ell/2}/2^{\ell} = 2c/\sqrt{t}$ for some constant $c > 0$.
  - $\mathbb{P}[A|s = 0] = 1$ since $s = 0 \Rightarrow r.z = r_j \Rightarrow A$.
  - $\mathbb{P}[A|s \neq 0] = 1/2$ since $s \neq 0 \Rightarrow s = \{\ldots, -4, -2, 2, 4, \ldots\}$. Hence, $\text{sign}(r.z) = \text{sign}(s)$ which is independent of r_j.

Proof of Lemma

Claim

Let A be the event $A = \{\text{sign}(r.z) = r_j\}$. For some constant $c > 0$,

$$\mathbb{P}[A] = \begin{cases} 1/2 & \text{if } z_j = 0 \\ 1/2 + c/\sqrt{t} & \text{if } z_j = 1 \end{cases}$$

- If $z_j = 0$: $\text{sign}(r.z)$ and r_j are independent so $\mathbb{P}[A] = 1/2$.
- If $z_j = 1$: Let $s = r.z - r_j$, the sum of an even number ($\ell = t/2 - 1$) of independent ± 1 values. Then,

$$\mathbb{P}[A] = \mathbb{P}[A|s = 0]\,\mathbb{P}[s = 0] + \mathbb{P}[A|s \neq 0]\,\mathbb{P}[s \neq 0]$$

  - $\mathbb{P}[s = 0] = \binom{\ell}{\ell/2}/2^\ell = 2c/\sqrt{t}$ for some constant $c > 0$.
  - $\mathbb{P}[A|s = 0] = 1$ since $s = 0 \Rightarrow r.z = r_j \Rightarrow A$.
  - $\mathbb{P}[A|s \neq 0] = 1/2$ since $s \neq 0 \Rightarrow s = \{\ldots, -4, -2, 2, 4, \ldots\}$. Hence, $\text{sign}(r.z) = \text{sign}(s)$ which is independent of r_j.
- So $\mathbb{P}[A] = \mathbb{P}[s = 0] + \frac{\mathbb{P}[s \neq 0]}{2} = \frac{1}{2} + \frac{\mathbb{P}[s = 0]}{2} = \frac{1}{2} + \frac{c}{\sqrt{t}}$.