Data Streams & Communication Complexity

Lecture 2: Graph Spanners, Sparsifiers, & Sketches

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Graph Streams

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$$\langle e_1, e_2, \ldots \quad \ldots, e_m \rangle$$

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  defining a graph G with nodes $V = [n]$ and $E = \{e_1, \ldots, e_m\}$
- *Semi-streaming:* What can we compute with $O(n \cdot \operatorname{polylog} n)$ space?

Outline

Spanners and Distances

Sparsifiers and Cuts

Sketches and Dynamic Graphs

- Connectivity
- k-Connectivity
- Minimum Cut

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Definition

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- *Thm:* Can count connected components in $O(n \log n)$ space.

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Theorem

Can $(2t - 1)$-approximate all distances using only $O(n^{1+1/t})$ space.

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- Because a) no cycles of length less than $2t + 1$ and b) all degrees in J are at least $d/2$, number of nodes at t-th level of BFS is at least

$$(d/2 - 1)^t = (m/n - 1)^t$$

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- But $(m/n - 1)^t \leq |J| \leq n$ and therefore $m \leq n + n^{1+1/t}$.

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Theorem (Batson, Spielman, Srivastava)

Exists offline algorithm $\mathcal{A}$ returning $(1+\epsilon)$-sparsifier with $O(n\epsilon^{-2})$ edges.

- *Idea:* Use $\mathcal{A}$ as a black box to recursively sparsify graph stream.

Basic Properties of Sparsifiers

Lemma

If H_1 and H_2 are α-sparsifiers of G_1 and G_2. Then $H_1 \cup H_2$ is an α-sparsifier of $G_1 \cup G_2$.

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Lemma

If J is an α-sparsifiers of H and H is an α-sparsifier of G. Then J is an α^2-sparsifier of G.

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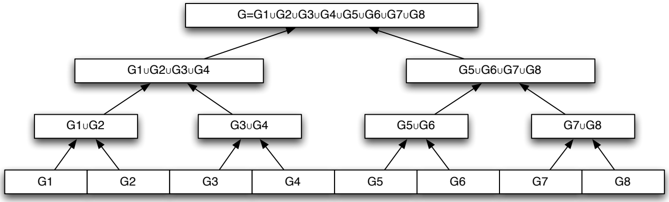

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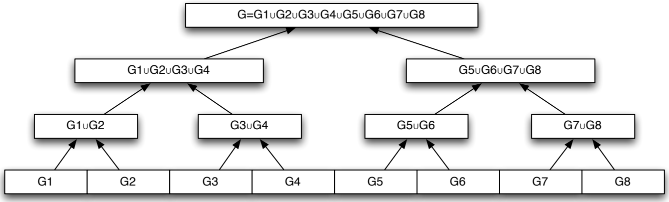

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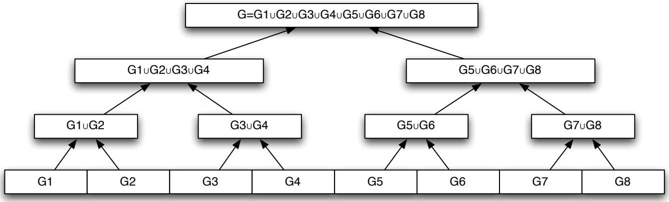

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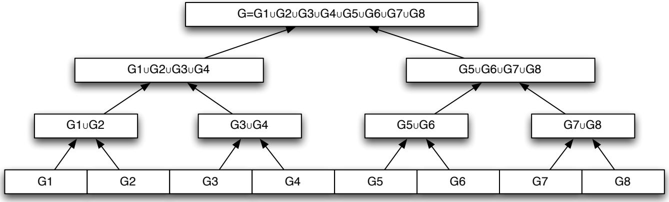

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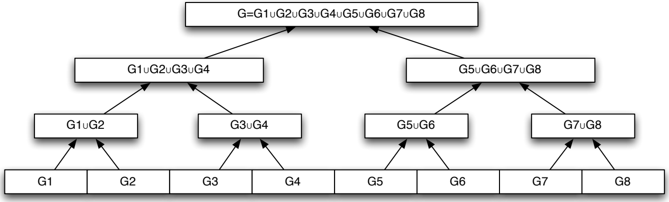

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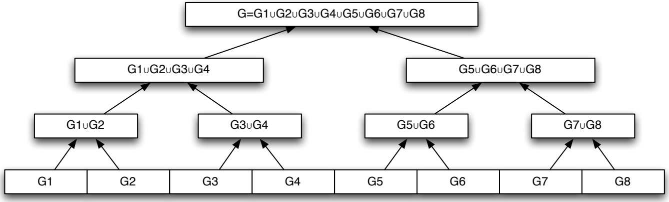

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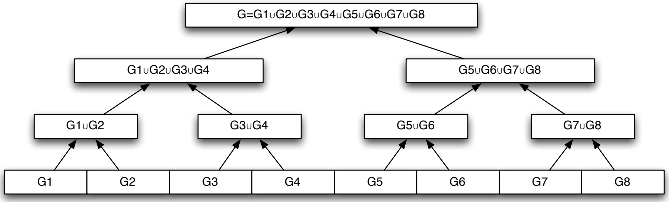

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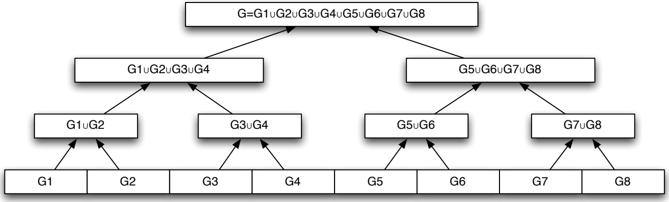

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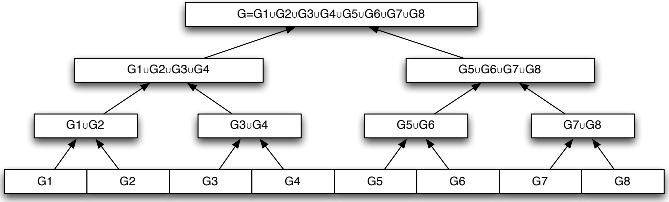

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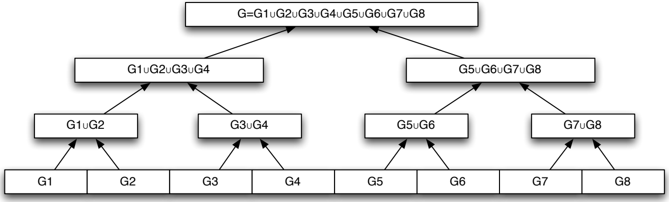

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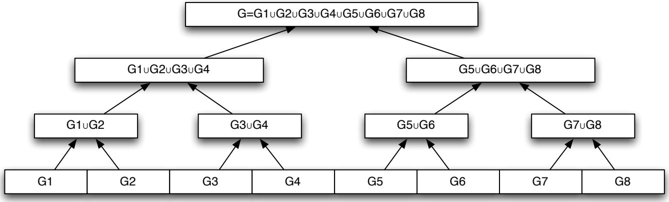

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- If $\gamma = O(\epsilon / \log m)$, we get $(1 + \epsilon)$-sparsifier in $O(n\epsilon^{-2} \log^3 m)$ space.

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- Consider a stream of edges inserts and deletions, e.g.,

  $\langle \texttt{add}(1,2), \texttt{add}(1,4), \texttt{add}(2,3), \texttt{add}(1,3), \texttt{add}(4,5), \texttt{add}(3,4), \texttt{del}(1,4) \rangle$

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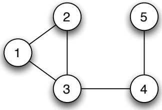

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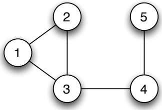

- *Dynamic semi-streaming:* What can we compute about a dynamic graph with only $O(n \cdot \text{polylog}\, n)$ space?

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Consider the offline algorithm:

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- *Idea:* Emulate above algorithm in a single pass using ℓ_0-sampling of a particular vector representation of G.

Useful Graph Representation

- Represent graph on $[n]$ with edges $E \subset [n] \times [n]$, as matrix
$$G \in \{-1, 0, 1\}^{n \times \binom{n}{2}}$$
with non-zero entries $G_{j,(j,k)} = 1$, $G_{k,(j,k)} = -1$ if $(j, k) \in E$.

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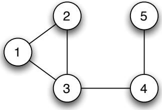

becomes,

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{c}
\begin{array}{cccccccccc} (1,2) & (1,3) & (1,4) & (1,5) & (2,3) & (2,4) & (2,5) & (3,4) & (3,5) & (4,5) \end{array} \\
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\
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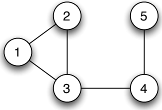

becomes,

$$\begin{array}{c c} & \begin{array}{cccccccccc} (1,2) & (1,3) & (1,4) & (1,5) & (2,3) & (2,4) & (2,5) & (3,4) & (3,5) & (4,5) \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \end{array}$$

- *Lemma:* For $S \subset [n]$, support$(\sum_{i \in S} a_i) = E(S)$ where a_i is ith row of A and $E(S)$ are edges across cut $(S, V \setminus S)$.

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- *Under-the-rug:* Actually we need to use $\log n$ independent sketch matrices $B, C, D, \ldots$ to emulate each round of algorithm. But this is fine: we can compute each $B(a_i), C(a_i), D(a_i), \ldots$ during same pass.

Outline

Spanners and Distances

Sparsifiers and Cuts

Sketches and Dynamic Graphs

- Connectivity
- k-Connectivity
- Minimum Cut

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Let F_1 be a spanning forest of G and, for $i \geq 2$, let F_i be a spanning forest of $G \setminus (F_1 \cup \ldots \cup F_{i-1})$. Then $F_1 \cup \ldots \cup F_k$ is a k-certificate for G.

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- *Idea:* Emulate above algorithm in a single pass by exploiting linearity of CONNECTIVITY algorithm.

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- And so on... resulting algorithm, $\textsc{connectivity}_k$, requires one pass and uses $O(k \cdot n \cdot \operatorname{polylog} n)$ space.

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Theorem (Karger)

Let $G = (V, E)$ be an unweighted graph with min-cut value λ. If we sample each edge with probability

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- *Idea:* Subsample the input graph at different rates and use $\textsc{connectivity}_k$ to compute min-cut size if it's small enough.

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- Can extend these ideas to get $(1 + \epsilon)$-sparsification of a dynamic graph in a single pass and $O(\epsilon^{-2} \cdot n \cdot \text{polylog}\, n)$ space.

Proof of Lemma

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- Consider a cut in G of size λ. Expected number of edges across same cut is G_i is at most

$$2p^* \cdot \lambda = 12\epsilon^{-2}\log n$$

and is $< \frac{24 \log n}{\epsilon^2} = k$ with high probability. Hence, $\lambda_i < k$.