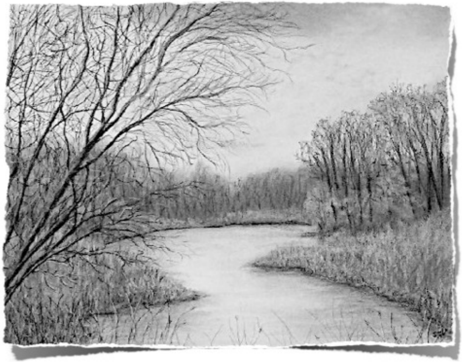
Data Streams & Communication Complexity

Lecture 1: Simple Stream Statistics in Small Space

Andrew McGregor, UMass Amherst



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- Origins in seventies but has become popular in last ten years...

Why's it become popular?

- *Practical Appeal:*
  - Faster networks, cheaper data storage, ubiquitous data-logging results in massive amount of data to be processed.
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- *Theoretical Appeal:*
  - Easy to state problems but hard to solve.
  - Links to communication complexity, compressed sensing, metric embeddings, pseudo-random generators, approximation...

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  - Frequency moments: $F_k = \sum_i f_i^k$.
  - Max frequency: $F_\infty = \max_i f_i$.
  - Number of distinct element: $F_0 = \sum_i f_i^0$
  - Median: j such that $f_1 + f_2 + \ldots + f_j \approx m/2$

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  Algorithms are often randomized and guarantees will be probabilistic.
- *Keep things simple:* Could consider f_i's being increased or decreased but for this talk we'll focus on unit increments. Will also assume algorithms have an unlimited store of random bits.

Outline

Sampling

Sketching: The Basics

Count-Min and Applications

Count-Sketch: Count-Min with a Twist

ℓ_p Sampling and Frequency Moments

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- *Beyond basic sampling:* There are more powerful forms of sampling and other techniques the make better use of the limited space.

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- *For high confidence:* Compute t estimators in parallel and average.

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$$\mathbb{P}\left[|\tilde{F}_k - F_k| \geq \epsilon F_k\right] \leq 2\exp\left(-\frac{tF_k\epsilon^2}{3kn^{1-1/k}F_k}\right) = 2\exp\left(-\frac{t\epsilon^2}{3kn^{1-1/k}}\right)$$

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- *Thm:* In $\tilde{O}(\epsilon^{-2}n^{1-1/k})$ space we can find a $(1 \pm \epsilon)$ approximation for F_k with probability at least $1 - \delta$.

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- *Useful:* Choose a distribution for $z_{i,j}$ such that relevant function of f can be estimated from s with high probability for sufficiently large w.

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- For the rest of lecture we'll focus on "hash-based" sketches. Given a random hash function $h : [n] \to [w]$, non-zero entries are $z_{h_i, i}$.

$$Z = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

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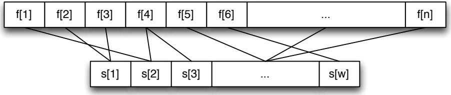

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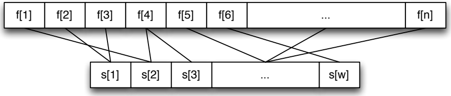

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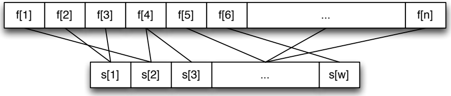

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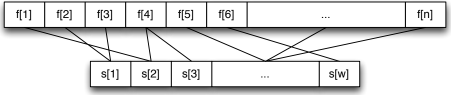

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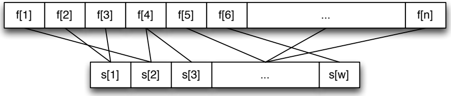

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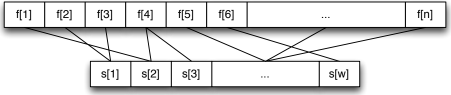

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- *Lemma:* $f_i \le \tilde{f}_i$ and $\mathbb{P}\left[\tilde{f}_i \ge f_i + 2m/w\right] \le 1/2$
- *Thm:* Let $w = 2/\epsilon$. Repeat the hashing $\lg(\delta^{-1})$ times in parallel and take the minimum estimate for f_i

$$\mathbb{P}\left[f_i \le \tilde{f}_i \le f_i + \epsilon m\right] \ge 1 - \delta$$

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- By an application of the Markov bound,

$$\mathbb{P}[\mathcal{E} \geq 2m/w] \leq 1/2$$

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- *Dyadic Intervals:* Restrict attention to intervals of the form

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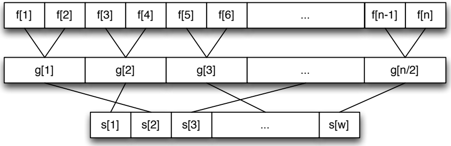
- E.g., for intervals of width 2 we have:



where update rule is now: for increment of f_{2i-1} or f_{2i}, increment s_{h_i}.

Quantiles and Heavy Hitters

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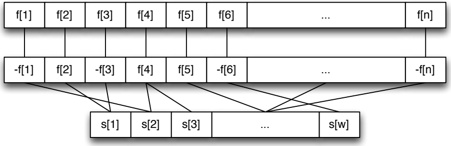
  Rather than checking each $\tilde{f}_i$ individually can save time by exploiting the fact that if $\tilde{f}_{[i,k]} < \phi m$ then $f_j < \phi m$ for all $j \in [i, k]$.

Outline

Count-Sketch: Count-Min with a Twist

- Maintain $s \in \mathbb{N}^w$ via hash functions $h : [n] \to [w]$, $r : [n] \to \{-1, 1\}$



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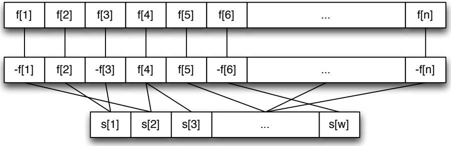

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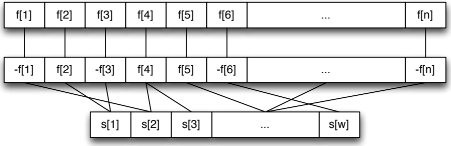

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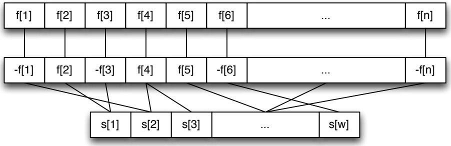

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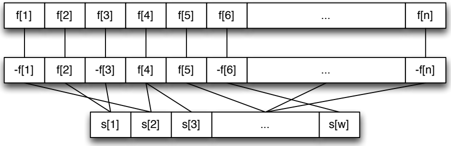

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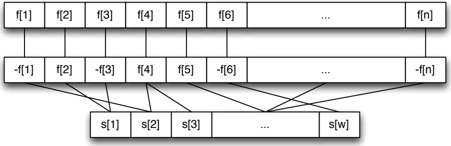

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- *Thm:* Let $w = O(1/\epsilon^2)$. Repeating $O(\lg \delta^{-1})$ in parallel and taking the median estimate ensures

$$\mathbb{P}\left[f_i - \epsilon\sqrt{F_2} \leq \tilde{f}_i \leq f_i + \epsilon\sqrt{F_2}\right] \geq 1 - \delta\ .$$

Proof of Lemma

- Define $\mathcal{E}$ by $\tilde{f}_i = f_i + \mathcal{E} r_i$ and so

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Outline

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- *ℓ_p Sampling:* Return random values $I \in [n]$ and $R \in \mathbb{R}$ where

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- Let's see algorithm for $p = 2 \ldots$

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- *Lemma:* Using a Count-Sketch of size $O(\epsilon^{-1} \log^2 n)$ ensures a $(1 \pm \epsilon)$ approximation of any g_i that passes the threshold.

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- *Under-the-rug:* Need to ensure that conditioning doesn't affect sampling probability too much.

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- Averaging over $t = O(\epsilon^{-2} n^{1-2/k} \log \delta^{-1})$ parallel repetitions gives,

$$\mathbb{P}\left[|\tilde{F}_k - F_k| \geq \epsilon F_k\right] \leq \delta$$

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- Earlier we used $O(n^{1-1/k})$ space to approximate $F_k = \sum_i |f_i|^k$.
- *Algorithm:* Let (I, R) be an $(1+\gamma)$-approximate ℓ_2 sample. Return

$$T = \tilde{F}_2 R^{k-2} \qquad \text{where } \tilde{F}_2 \text{ is a } (1 \pm \gamma) \text{ approximation for } F_2$$

- *Expectation:* Setting $\gamma = \epsilon/(4k)$,

$$\mathbb{E}[T] = \tilde{F}_2 \sum \mathbb{P}[I = i]((1\pm\gamma)f_i)^{k-2} = (1\pm\gamma)^k F_2 \sum \frac{f_i^2}{F_2} f_i^{k-2} = (1\pm\frac{\epsilon}{2})F_k$$

- *Range:* $0 \leq T \leq (1+\gamma)F_2 F_\infty^{k-2} = (1+\gamma)n^{1-2/k}F_k$.
- Averaging over $t = O(\epsilon^{-2}n^{1-2/k}\log\delta^{-1})$ parallel repetitions gives,

$$\mathbb{P}\left[|\tilde{F}_k - F_k| \geq \epsilon F_k\right] \leq \delta$$

- *Thm:* In $\tilde{O}(\epsilon^{-2}n^{1-2/k})$ space we can find a $(1\pm\epsilon)$ approximation for F_k with probability at least $1-\delta$.

Summary

- *Basic Sampling:* Can do basic sampling where i is selected with probability $\propto f_i$ but we can be much smarter via sketches.
- *Count-Min:* $f_i \leq \tilde{f}_i \leq f_i + \epsilon F_1$ in $O(\epsilon^{-1})$ space.

$$Z = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

- *Count-Sketch:* $f_i - \epsilon\sqrt{F_2} \leq \tilde{f}_i \leq f_i + \epsilon\sqrt{F_2}$ in $O(\epsilon^{-2})$ space.

$$Z = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

  Above sketches solve range-queries, quantiles, heavy hitters, ...
- *ℓ_p-Sampling:* Selecting i with probability $\propto f_i^p$ in $O(\epsilon^{-2})$ space.

$$Z = \begin{bmatrix} 0 & \gamma_2 & 0 & 0 & 0 & -\gamma_6 \\ 0 & 0 & 0 & -\gamma_4 & 0 & 0 \\ -\gamma_1 & 0 & \gamma_3 & 0 & \gamma_5 & 0 \end{bmatrix}$$