Homomorphic Sketches

Shrinking Big Data without Sacrificing Structure

Andrew McGregor
University of Massachusetts

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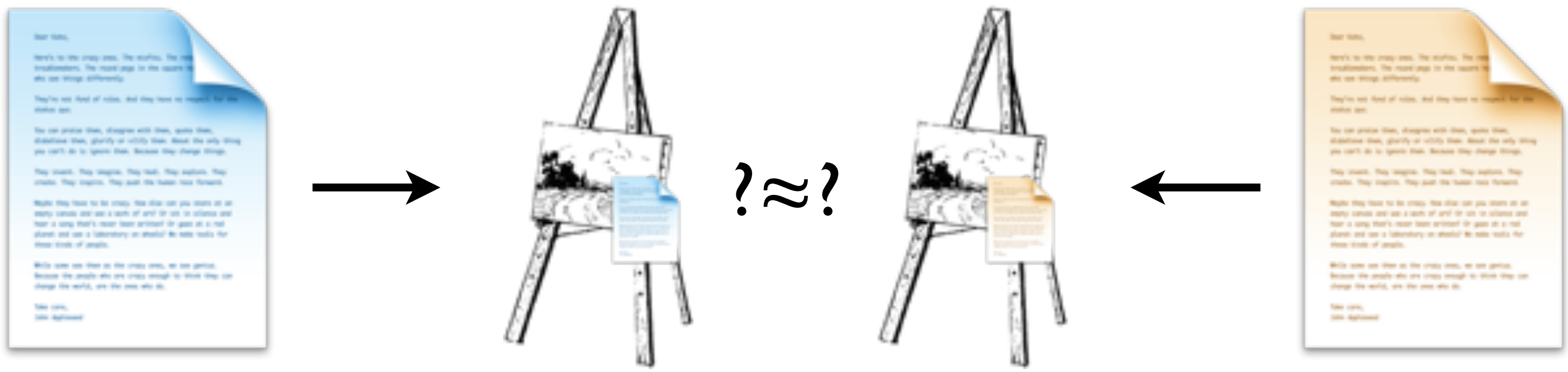
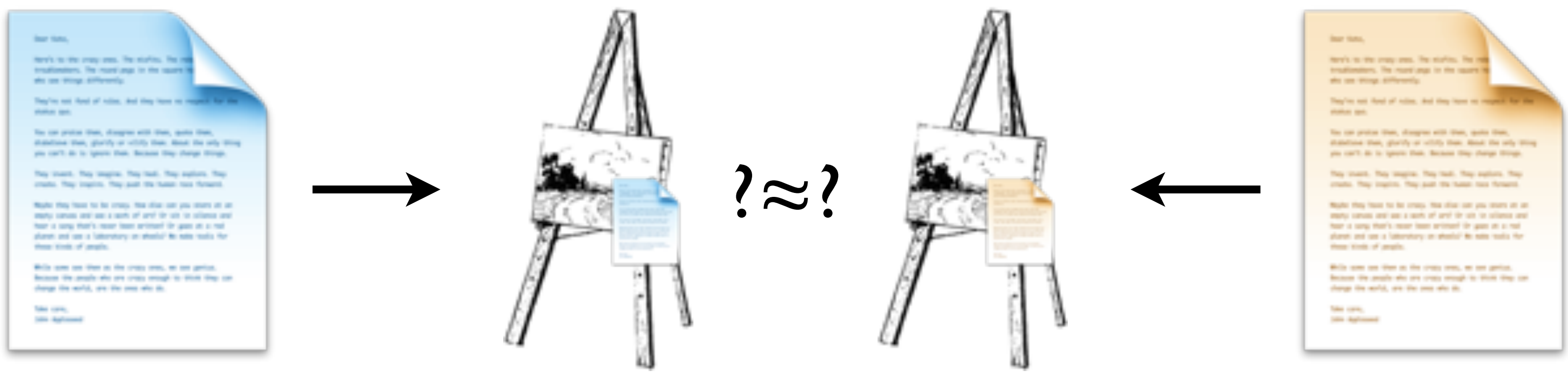


More generally, can construct *sketches* of files to estimate Hamming distance between the files.



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Many results such as distinct elements, entropy, frequency moments, quantiles, histograms, linear regression, clustering, shape approximation...

Basic Idea: Treat file as vector; use *linear projections* to reduce dimension while preserving properties.

Extensive theory with connections to compressed sensing, metric embeddings; *widely applicable* since parallelizable and suitable for stream processing.

Most existing work concerns numerical statistics of data such as frequency and feature vectors...

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BIG DATA

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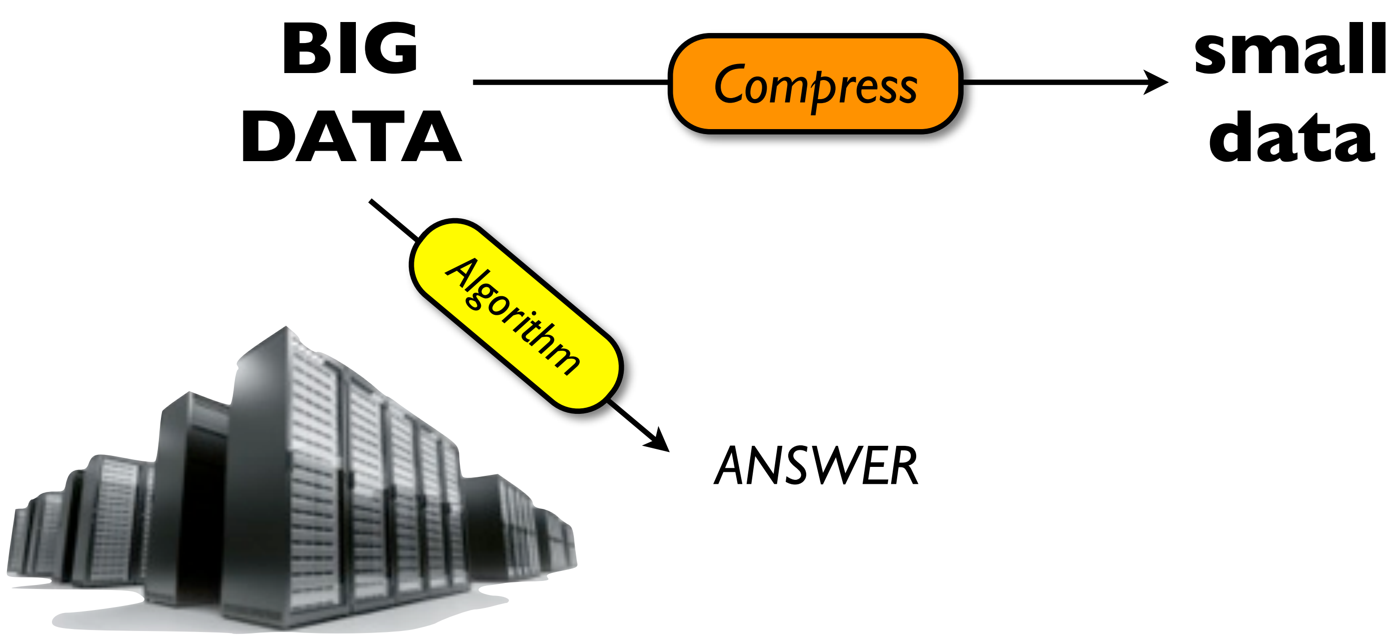

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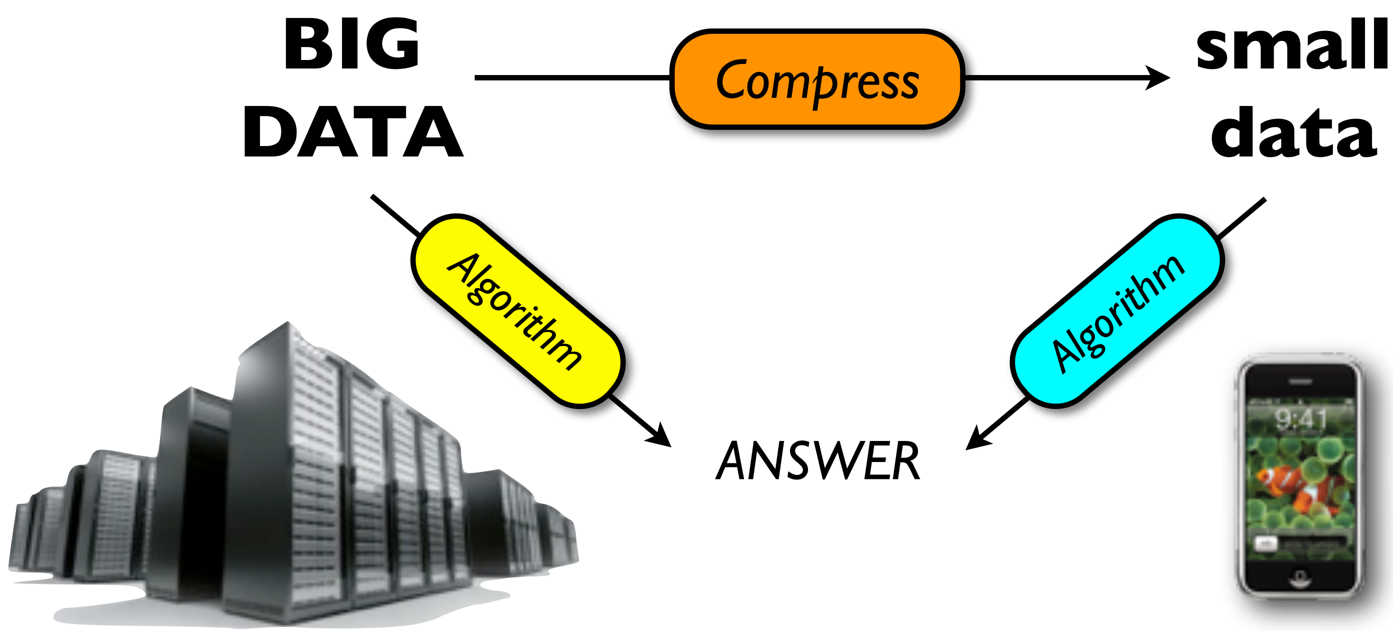

First Result...

Problem: Fingerprint each row of nxn adjacency matrix such that we can check connectivity using fingerprints.

Theorem: Fingerprints of size O(polylog n) bit suffice!

Surprising? Seems impossible if there are bridge edges.

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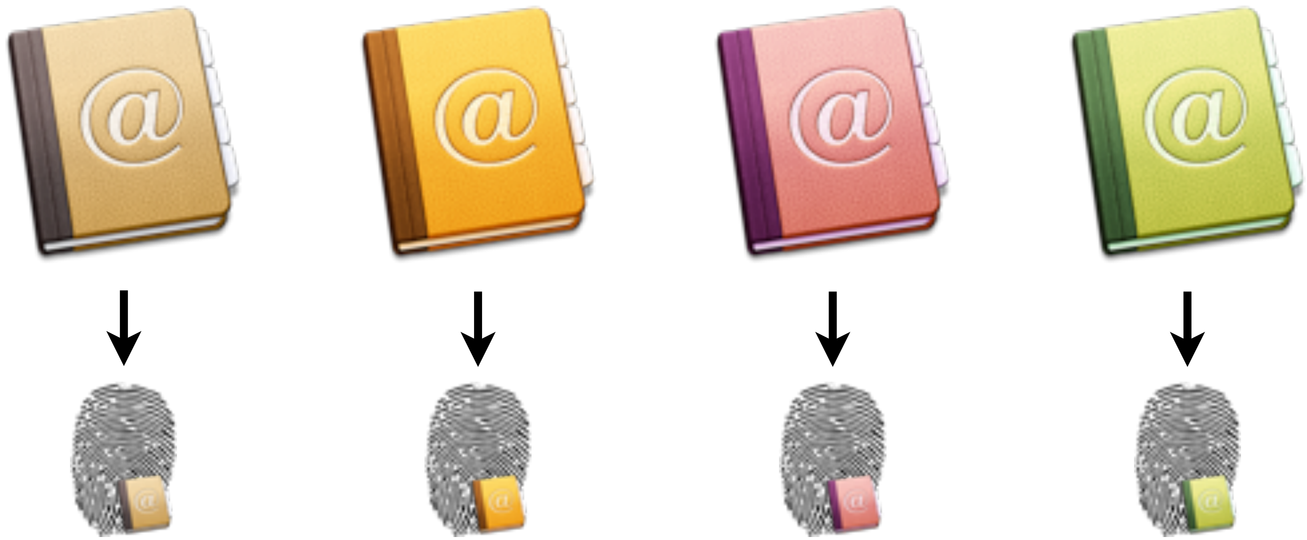

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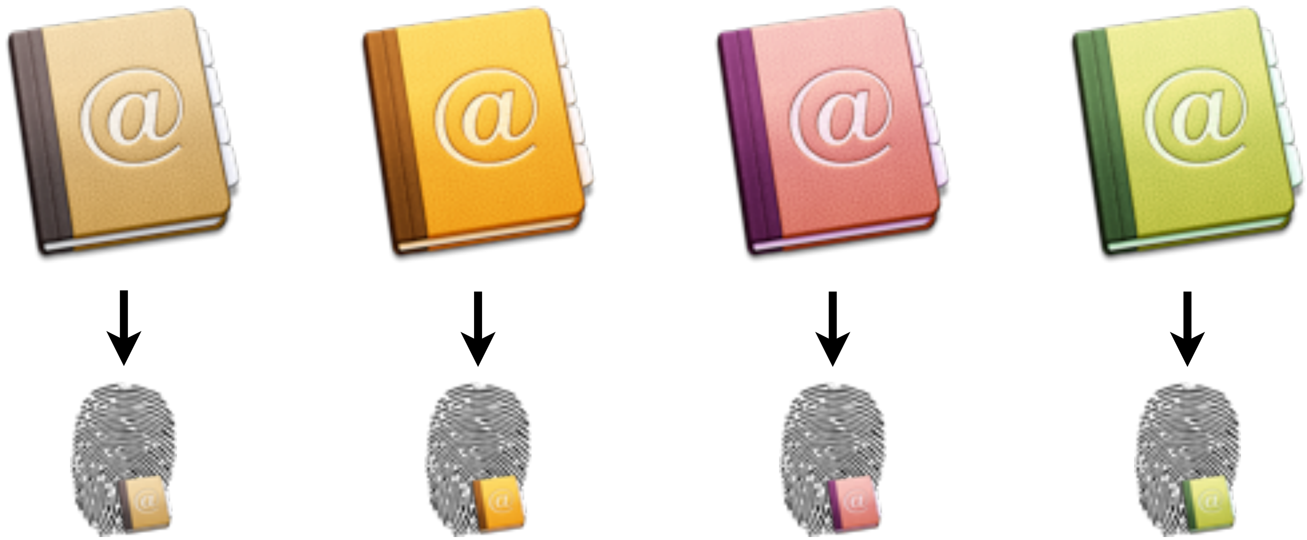

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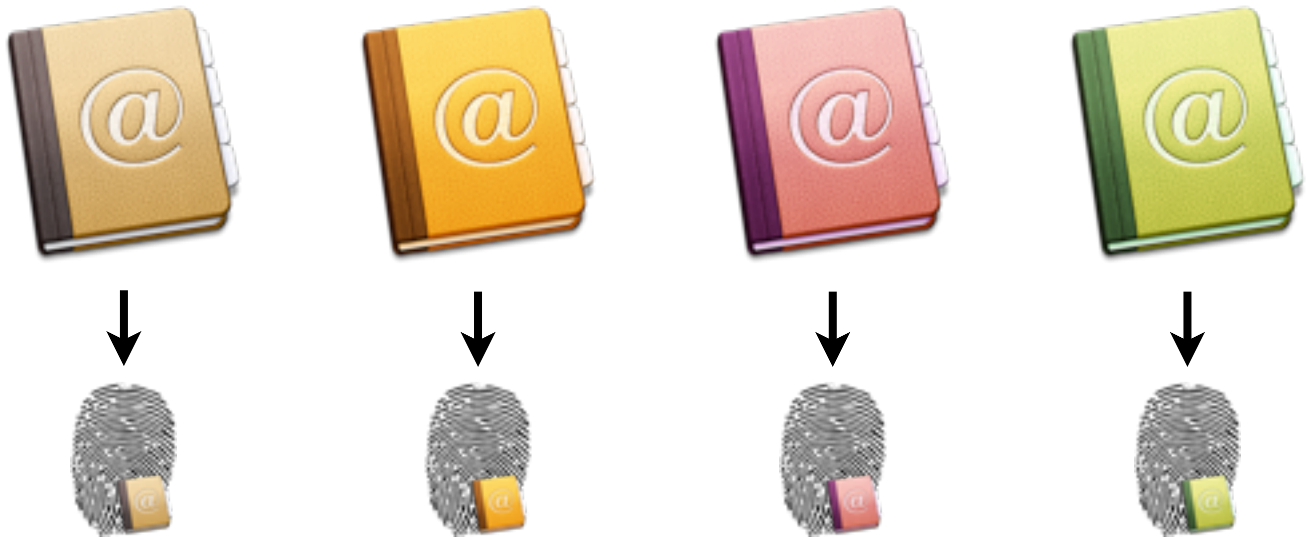

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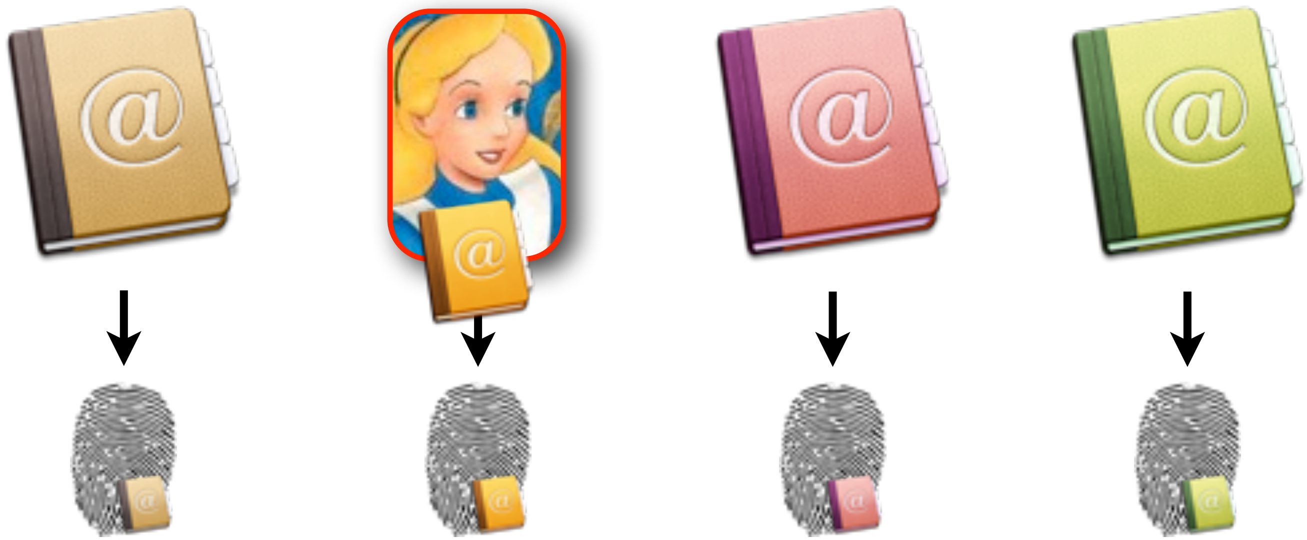

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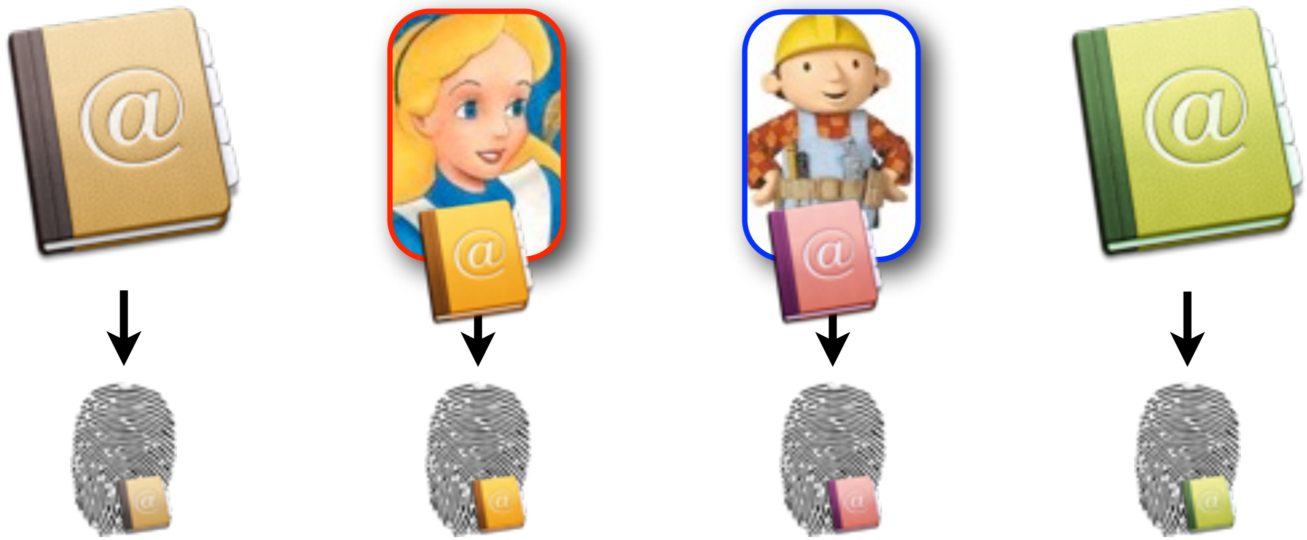

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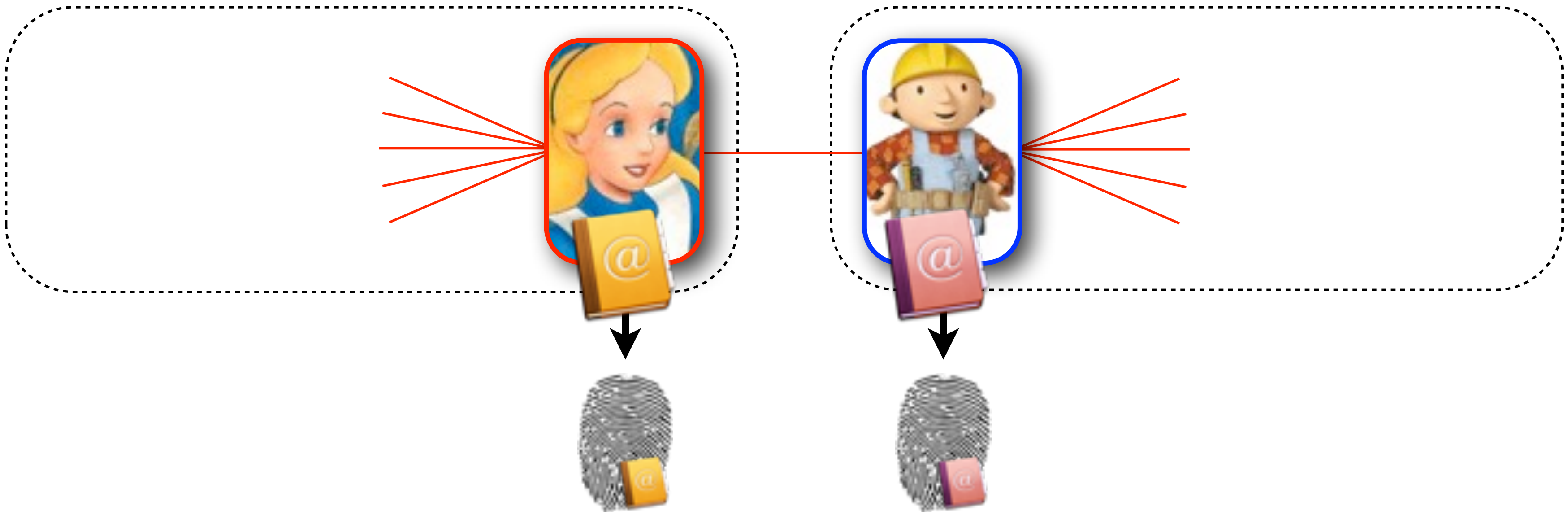

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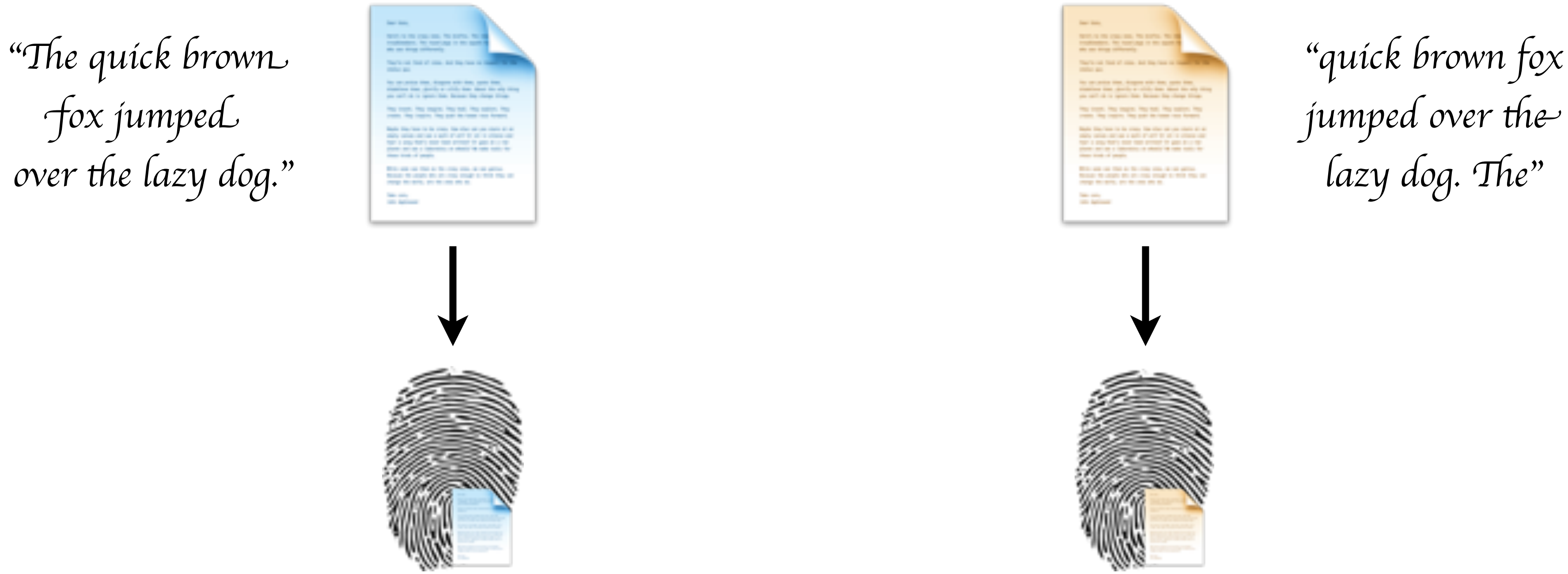

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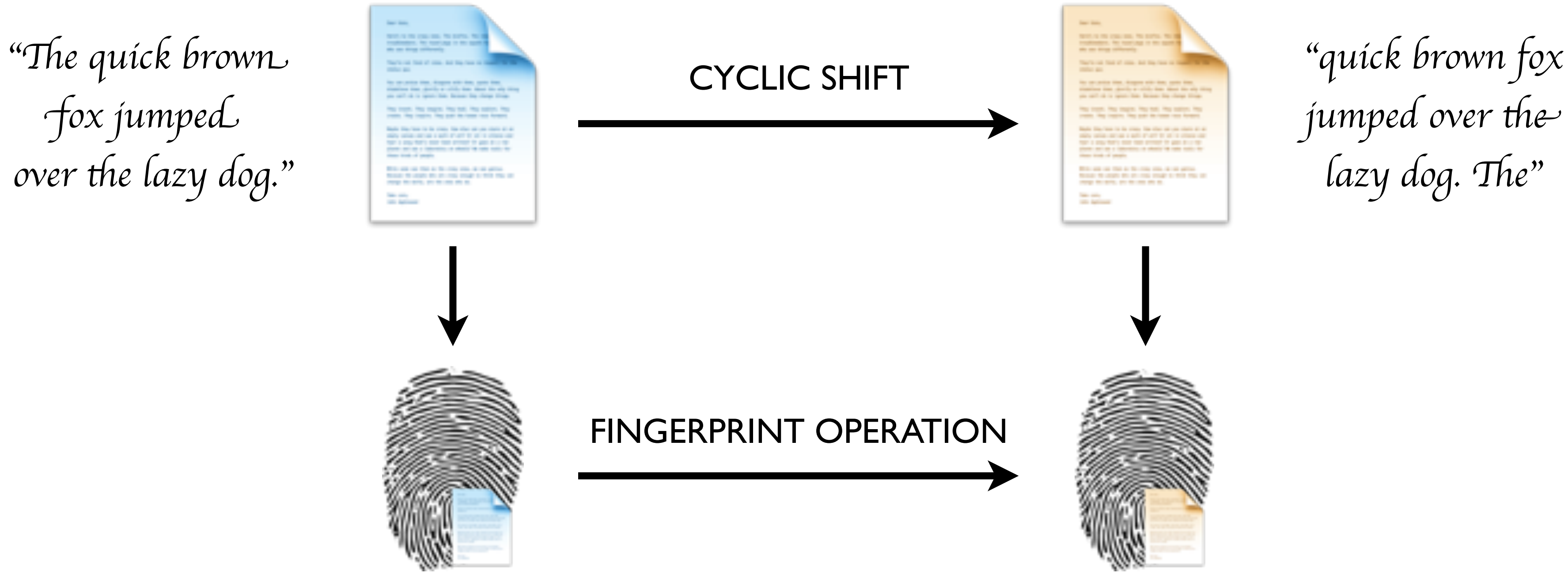

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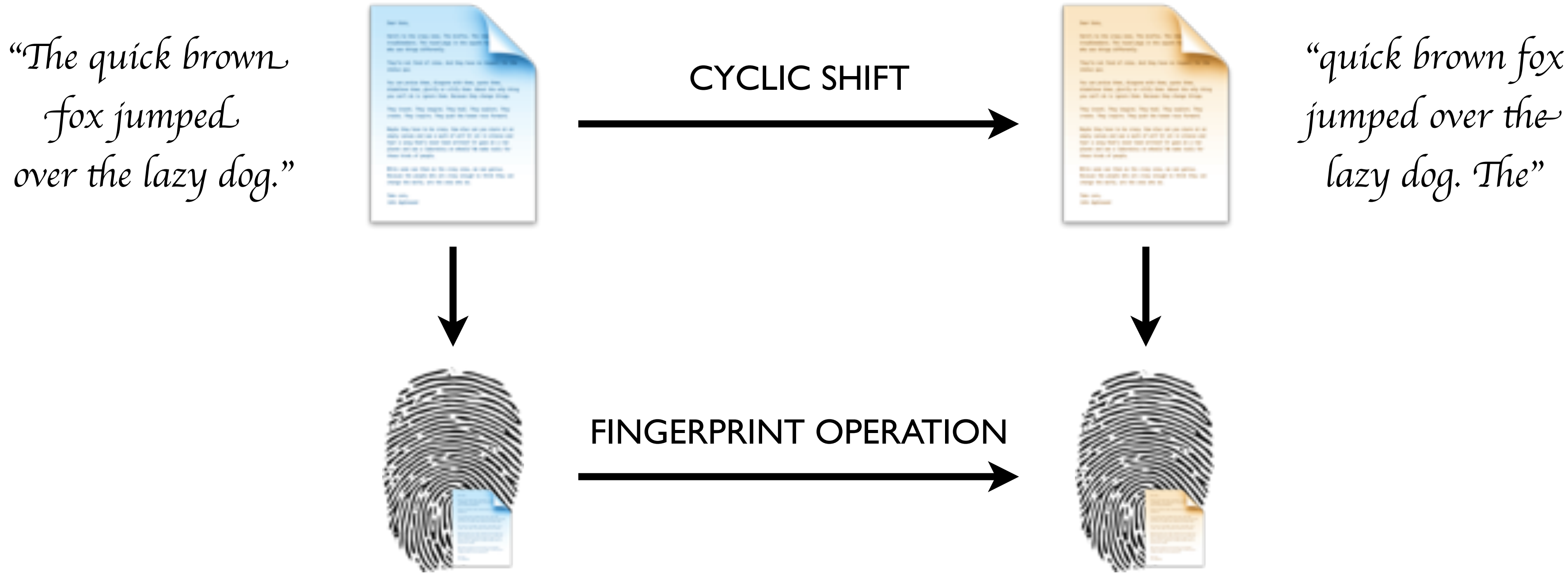

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Surprising? Fingerprint size isn't monotonic in file size!

I. **Connectivity**

II. **Misalignment**

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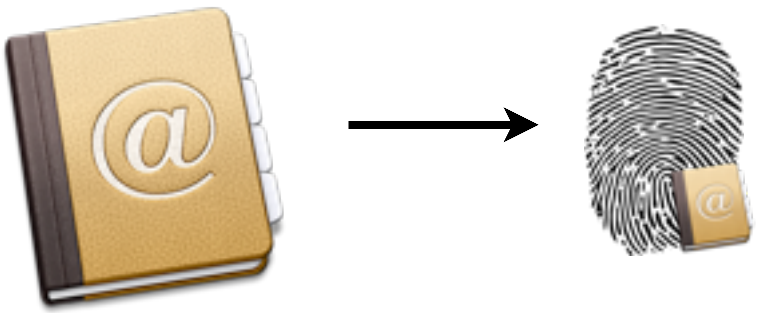
a) Connectivity via O(polylog n) bit Fingerprints

b) Extension to Estimating Cuts and Eigenvalues

Joint work with Kook Jin Ahn and Sudipto Guha

Sketches for Connectivity



- *Theorem:* Can check k-connectivity w.h.p. using O(k polylog n) bit fingerprint of each adjacency list.

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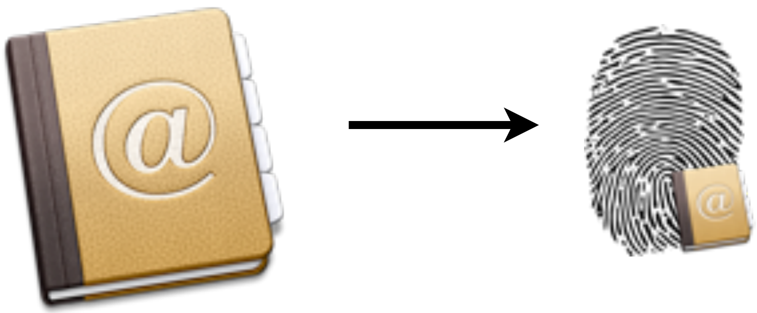

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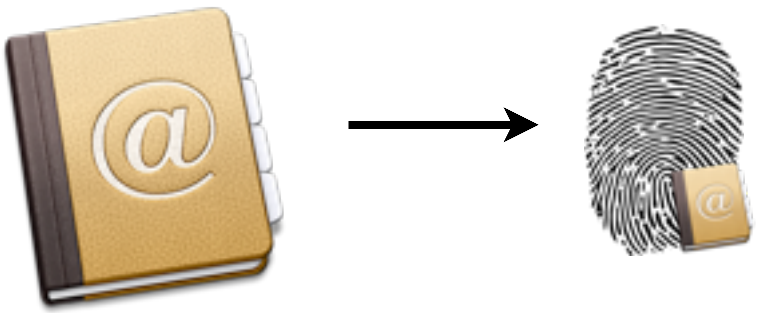

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e.g., [Feigenbaum, Kannan, McGregor, Suri, Zhang 2004, 2005], [McGregor 2005] [Jowhari, Ghodsi 2005], [Zelke 2008], [Sarma, Gollapudi, Panigrahy 2008, 2009] [Ahn, Guha 2009, 2011], [Konrad, Magniez, Mathieu 2012], [Goel, Kapralov, Khanna 2012]

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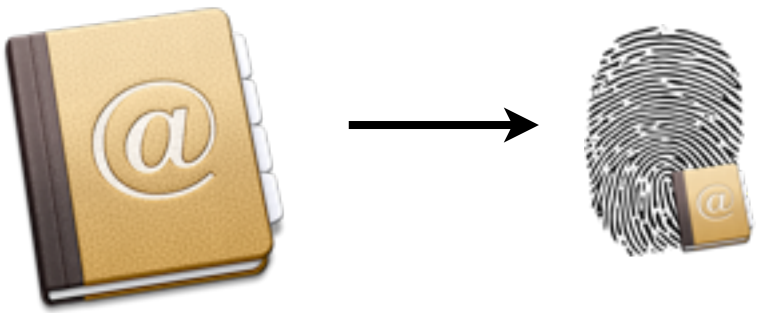

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- New sliding window graph results presented yesterday.

[Crouch, McGregor, Stubbs 2013]

Basic Primitive: Neighborhood Sketches

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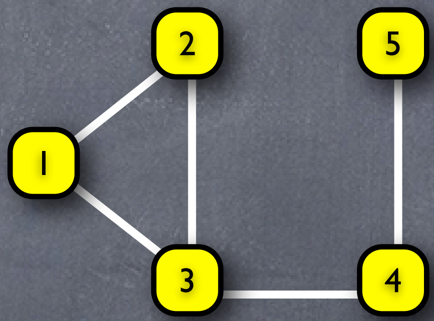

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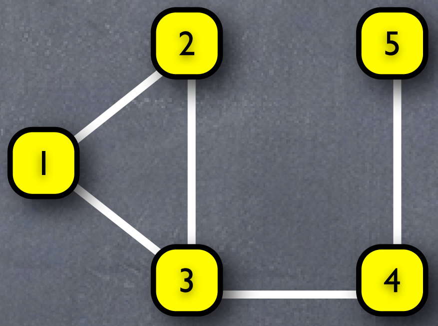

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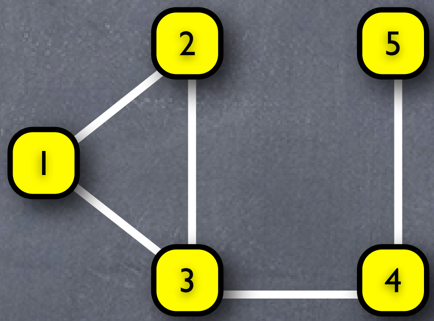

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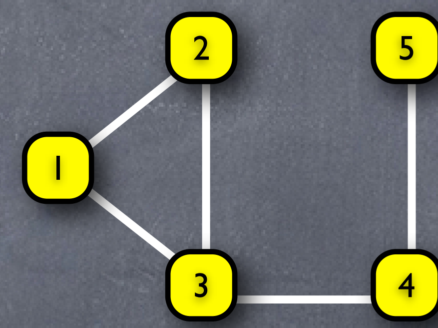

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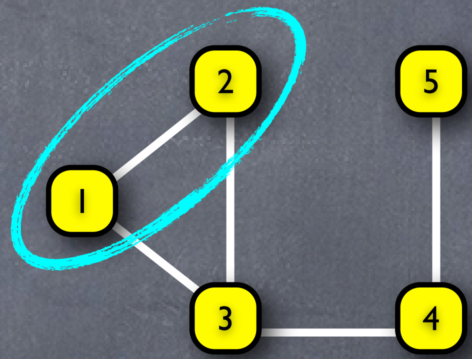

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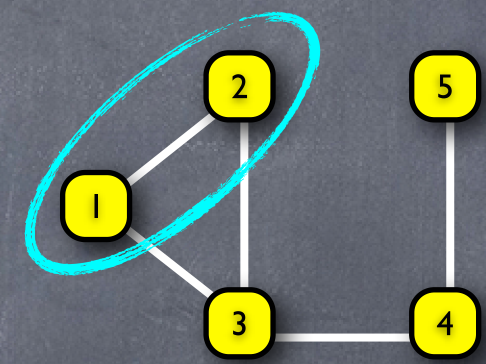

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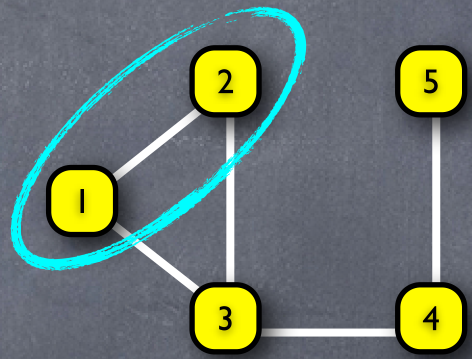

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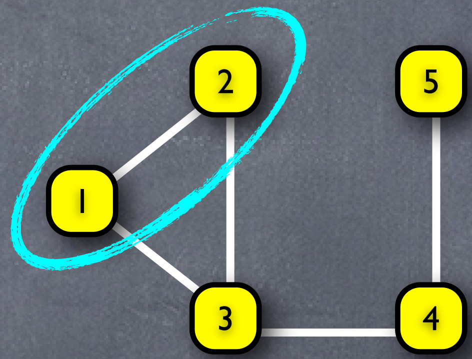

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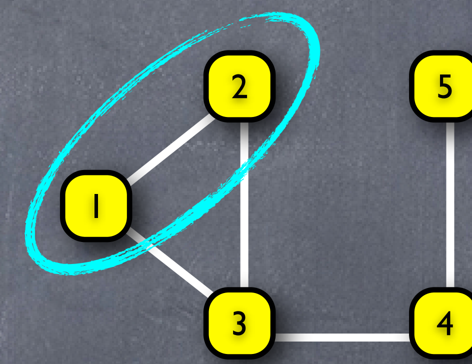

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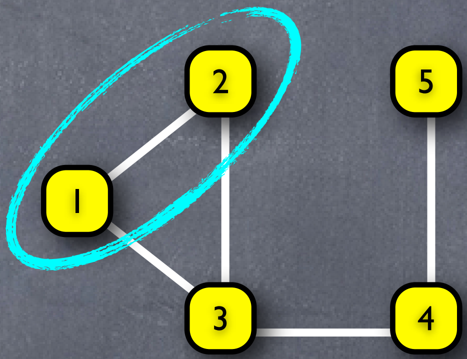

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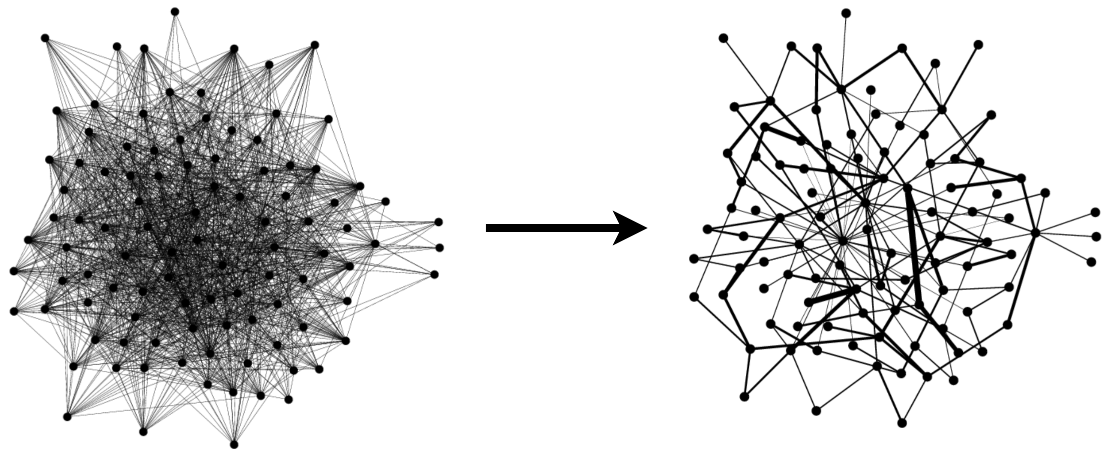
Extension to Sparsification



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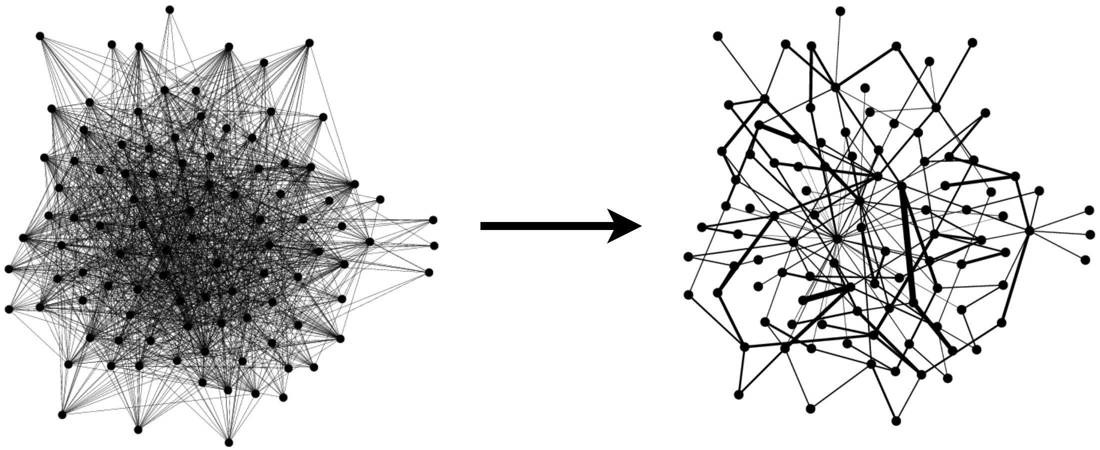

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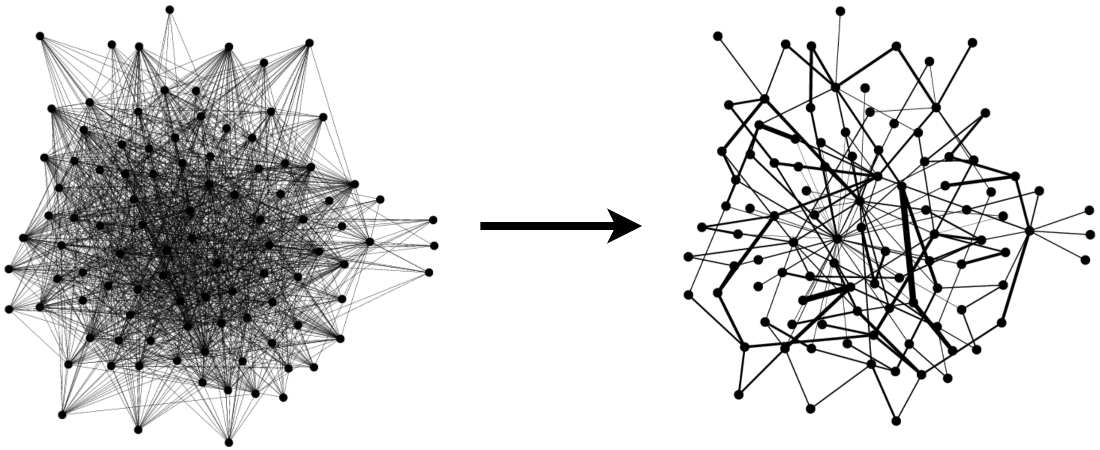
- *Theorem:* Can $(1+\varepsilon)$-approximate every graph cut using $O(\varepsilon^{-2}$ polylog n) bit fingerprints of each adjacency list.

Extension to Sparsification



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- *Theorem:* Can (1+ε)-approximate every graph cut using $O(\varepsilon^{-2}$ polylog n) bit fingerprints of each adjacency list.
- *Theorem:* Can construct a spectral sparsifier H using $O(\varepsilon^{-2} n^{2/3}$ polylog n) bit fingerprints of each adjacency list.

$$\forall x \in \mathbb{R}^n \ : \ (1-\epsilon)x^T L_G x \le x^T L_H x \le (1+\epsilon)x^T L_G x$$

where L_G and L_H are the Laplacians of G and H.

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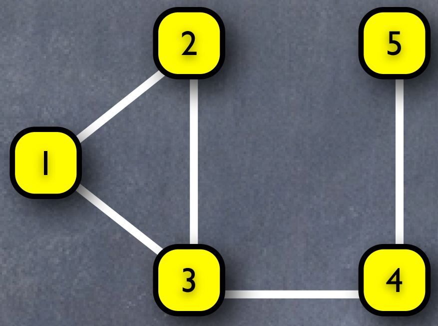

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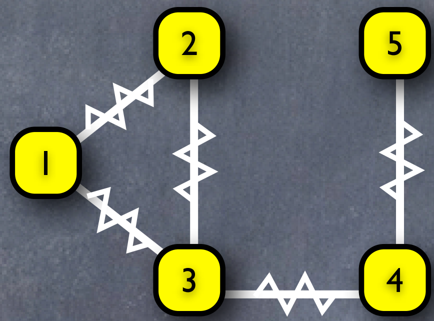

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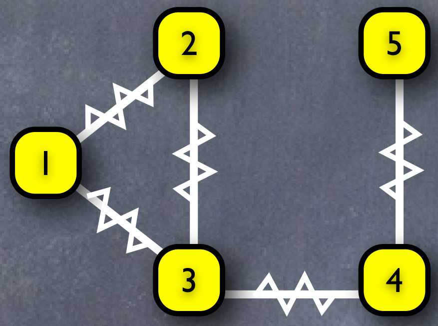

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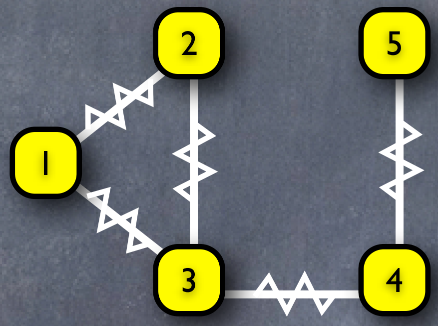

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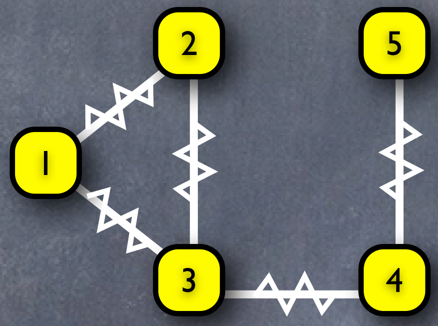

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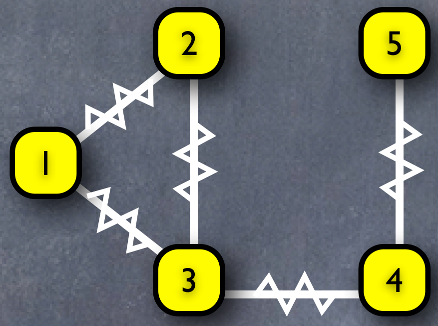

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- Corollary: Increasing sampling probability by $O(n^{2/3})$ in cut sparsification, also preserves spectral properties.

I. **Connectivity**

II. **Misalignment**

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a) Testing Equality with Rotation

b) Matching Lower Bound

Joint work with Alexandr Andoni, Assaf Goldberger, Ely Porat

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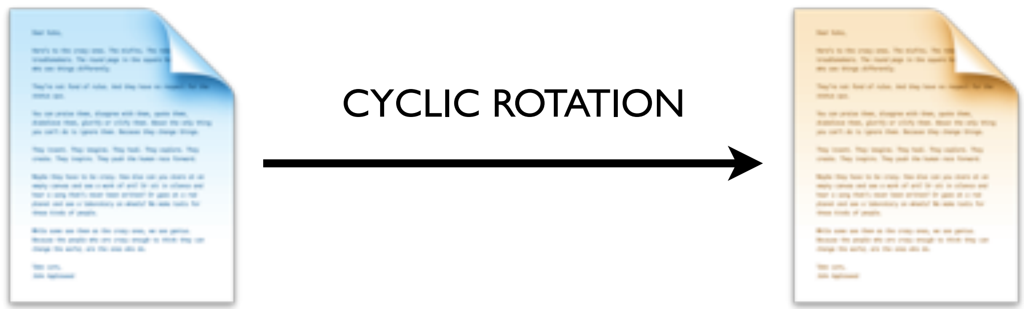

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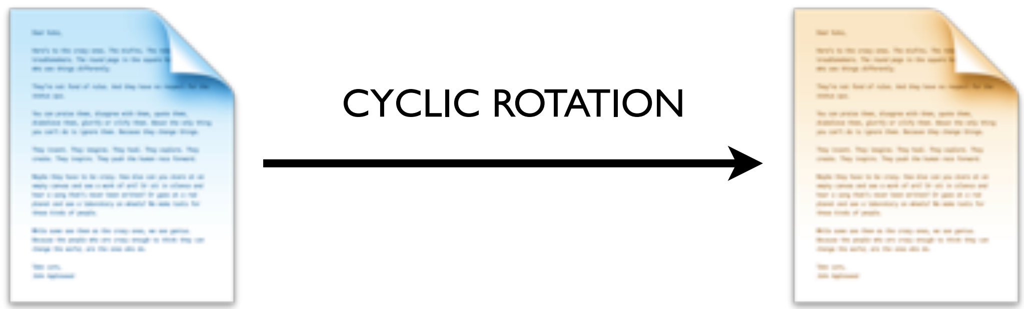

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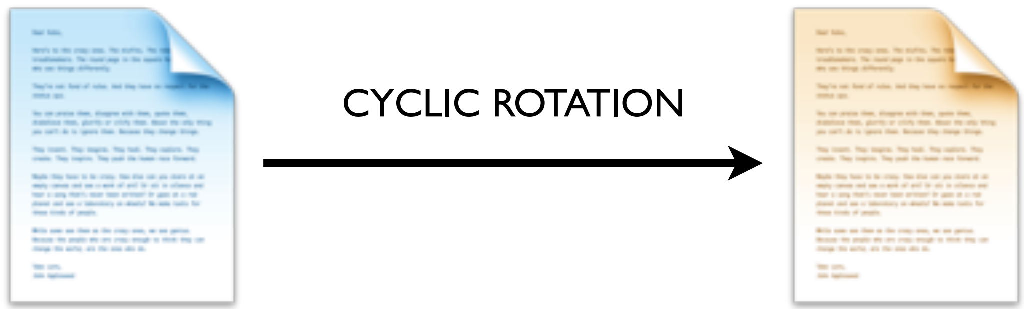

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CYCLIC ROTATION

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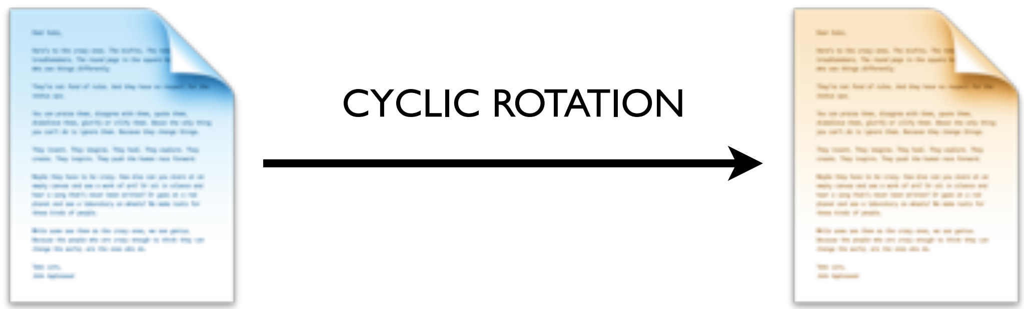

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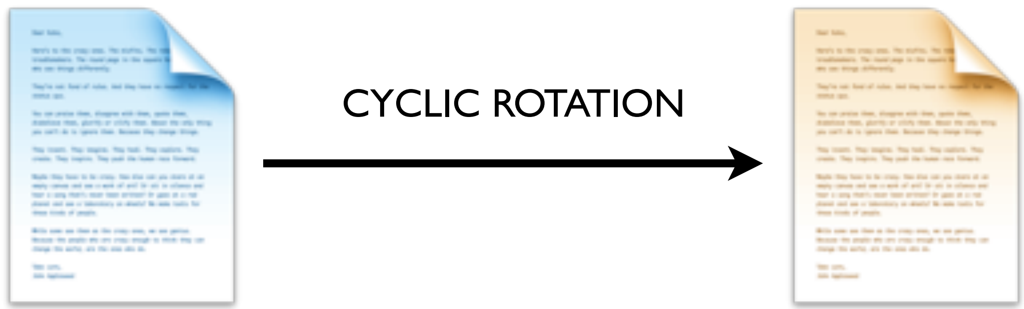

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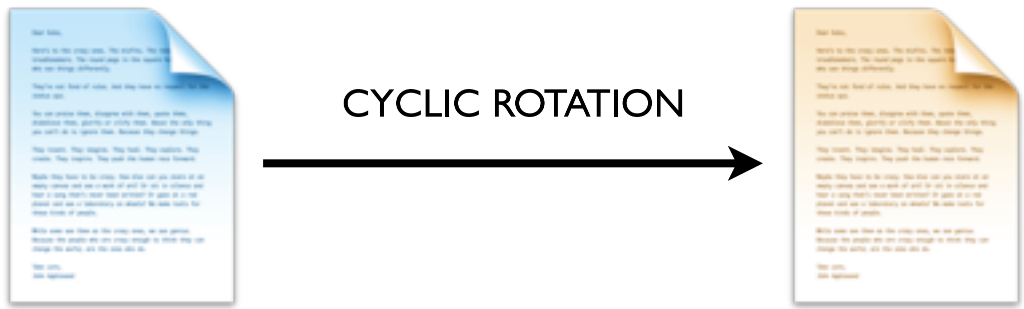

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- Conclusion: No false negatives but likely false positives.

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- Good News: Work modulo a random p. Can show ϕ_i still doesn't share roots with g whp by analyzing resultant.

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and compare $F(gggggg)$ for all g until matches.

- To deduce $\beta = a_1 + a_3 + a_5$

$$F(a_0 a_1 a_2 a_3 a_4 a_5) + F(a_2 a_3 a_4 a_5 a_0 a_1) + F(a_4 a_5 a_0 a_1 a_2 a_3) = F(\beta\gamma\beta\gamma\beta\gamma)$$

Lower Bound: Basic Idea

- Can recover D(n) bits about a from F(a) by summing the fingerprints of rotations
- To deduce $\alpha = \sum a_i$ from $F(a_0 a_1 a_2 a_3 a_4 a_5)$

$$F(a_0 a_1 a_2 a_3 a_4 a_5) + F(a_1 a_2 a_3 a_4 a_5 a_0) + \ldots + F(a_5 a_0 a_1 a_2 a_3 a_4) = F(\alpha\alpha\alpha\alpha\alpha\alpha)$$

and compare $F(gggggg)$ for all g until matches.

- To deduce $\beta = a_1 + a_3 + a_5$

$$F(a_0 a_1 a_2 a_3 a_4 a_5) + F(a_2 a_3 a_4 a_5 a_0 a_1) + F(a_4 a_5 a_0 a_1 a_2 a_3) = F(\beta\gamma\beta\gamma\beta\gamma)$$

and compare $F(gg'gg'gg')$ for all g, g'=α-g until matches.

Lower Bound: Basic Idea

- Can recover D(n) bits about a from F(a) by summing the fingerprints of rotations
- To deduce $\alpha = \sum a_i$ from $F(a_0 a_1 a_2 a_3 a_4 a_5)$

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and compare $F(gggggg)$ for all g until matches.

- To deduce $\beta = a_1 + a_3 + a_5$

$$F(a_0 a_1 a_2 a_3 a_4 a_5) + F(a_2 a_3 a_4 a_5 a_0 a_1) + F(a_4 a_5 a_0 a_1 a_2 a_3) = F(\beta\gamma\beta\gamma\beta\gamma)$$

and compare $F(gg'gg'gg')$ for all g, g'=α-g until matches.

- And so on for other divisors of n...

Thanks!

- *Homomorphic Sketches:* Compress using sketches such that we can run algorithms on compressed data directly. Resulting algorithms are *parallelizable* + *streamable*.
- *Graphs:* Dimensionality reduction for preserving structural properties. Enables dynamic graph streaming.
- *Fingerprinting with Misalignments:* Tight bounds on size of fingerprint necessary for testing equality up to rotations.