Lecture 9: PSPACE

$$\text{PSPACE} \quad = \quad \text{DSPACE}[n^{O(1)}] \quad = \quad \text{NSPACE}[n^{O(1)}] \quad = \quad \text{ATIME}[n^{O(1)}]$$

- PSPACE consists of what we could compute with a feasible amount of hardware, but with no time limit.
- PSPACE is a large and very robust complexity class.
- With polynomially many bits of memory, we can search any implicitly-defined graph of exponential size. This leads to complete problems such as reachability on exponentially-large graphs.
- We can search the game tree of any board game whose configurations are describable with polynomially-many bits and which lasts at most polynomially many moves. This leads to complete problems concerning winning strategies.

PSPACE-Complete Problems

Recall PSPACE $=$ ATIME$[n^{O(1)}]$

Recall QSAT, the quantified satisfiability problem.

Proposition 9.1 QSAT *is* PSPACE-*complete.*

Proof: We've already seen that QSAT $\in$ ATIME$[n]$ $\subseteq$ PSPACE.

QSAT is hard for ATIME$[n^k]$:

Let M be an ATIME$[n^k]$ TM, w an input, $n = |w|$

Let M write down its n^k alternating choices, $c_1\ c_2 \ldots c_{n^k}$.

Deterministic TM D evaluates the answer, i.e., for all inputs w, $M(w) = 1 \Leftrightarrow \exists c_1 \forall c_2 \cdots \exists c_{n^k} (D(\bar{c}, w) = 1)$

By Cook's Theorem $\exists$ reduction $f : \mathcal{L}(D) \leq$ SAT:

$$D(\bar{c}, w) = 1 \quad \Leftrightarrow \quad f(\bar{c}, w) \in \text{SAT}$$

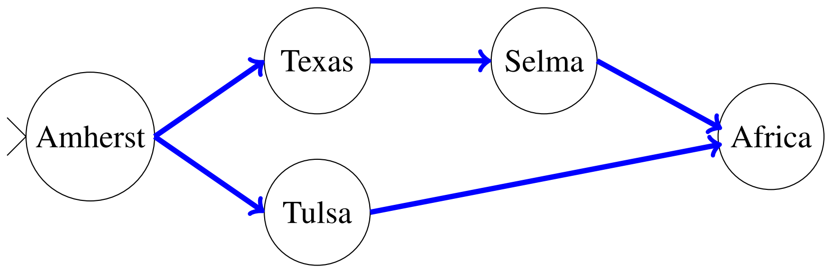
Let the new boolean variables in $f(\bar{c}, w)$ be $d_1 \ldots d_{t(n)}$.

$$M(w) = 1 \quad \Leftrightarrow \quad \text{“}\exists c_1 \forall c_2 \cdots \exists c_{n^k}\, d_1 \ldots d_{t(n)}\, (f(\bar{c}, w))\text{”} \in \text{QSAT} \qquad \square$$

Geography is a two-person game.

1. E "chooses" the start vertex v_1.
2. A chooses v_2, having an edge from v_1
3. E chooses v_3, having an edge from v_2, etc.

No vertex may be chosen twice. Whoever moves last wins.



Let GEOGRAPHY be the set of positions in geography games s.t. $\exists$ has a winning strategy.

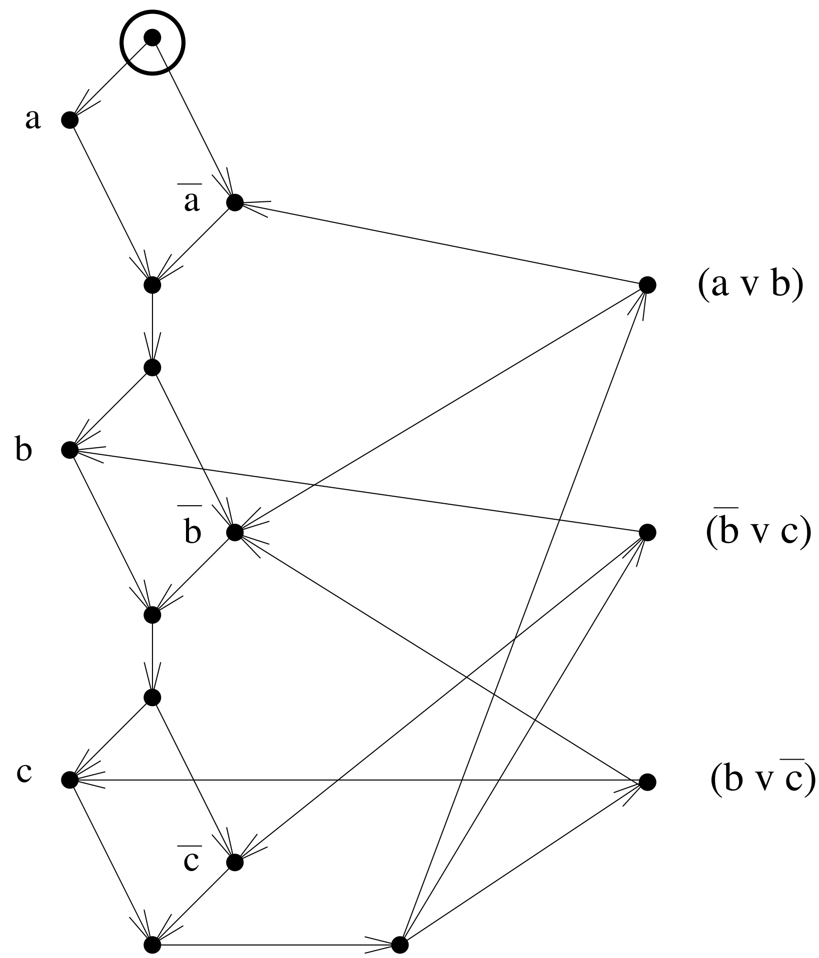
Proposition 9.2 GEOGRAPHY *is* PSPACE-*complete.*

Proof: GEOGRAPHY $\in$ PSPACE: search the polynomial-depth game tree. A polynomial-size stack suffices.

Show: QSAT $\leq$ GEOGRAPHY

Given formula, φ, build graph G_φ s.t. $\exists$ chooses existential variables; $\forall$ chooses universal variables.

$$\begin{aligned}\varphi \quad \equiv \quad & \exists a \, \forall b \, \exists c \\ & [(a \vee b) \ \wedge \\ & (\bar{b} \vee c) \ \wedge \\ & (b \vee \bar{c})]\end{aligned}$$



□

Definition 9.3 A **succinct** representation of a graph is $G(n, C, s, t) = (V, E, s, t)$ where C is a boolean circuit with $2n$ inputs and

$$\begin{aligned} V &= \{w \mid w \in \{0,1\}^n\} \\ E &= \{(w, w') \mid C(w, w') = 1\} \end{aligned}$$

□

$$\text{SUCCINCT REACH} = \{(n, C, s, t) \mid G(n, C, s, t) \in \text{REACH}\}$$

Proposition 9.4 SUCCINCT REACH $\in$ PSPACE

Why?

Remember Savitch's Thm:

$$\text{REACH} \in \text{NSPACE}[\log n] \subseteq \text{DSPACE}[(\log n)^2]$$

$$\text{SUCCINCT REACH} \in \text{NSPACE}[n] \subseteq \text{DSPACE}[n^2] \subseteq \text{PSPACE}$$

□

Proposition 9.5 SUCCINCT REACH *is* PSPACE-*complete.*

Proof: Let M be a DSPACE$[n^k]$ TM, input w, $n = |w|$

$$M(w) = 1 \quad \leftrightarrow \quad \text{CompGraph}(M, w) \in \text{REACH}$$

$$\text{CompGraph}(n, w) \quad = \quad (V, E, s, t)$$

$$\begin{aligned} V &= \{\text{ID} = \langle q, h, p\rangle \mid q \in \text{States}(N), h \leq n, |p| \leq c\,n^k\} \\ E &= \{(\text{ID}_1, \text{ID}_2) \mid \text{ID}_1(w) \underset{M}{\longrightarrow} \text{ID}_2(w)\} \\ s &= \text{initial ID} \\ t &= \text{accepting ID} \end{aligned}$$

□

Succinct Representation of CompGraph(n, w):

$$
\begin{aligned}
V &= \left\{ \text{ID} = \langle q, h, p \rangle \;\middle|\; q \in \text{States}(N), h \le n, |p| \le c\, n^k \right\} \\
E &= \left\{ (\text{ID}_1, \text{ID}_2) \;\middle|\; \text{ID}_1(w) \underset{M}{\longrightarrow} \text{ID}_2(w) \right\}
\end{aligned}
$$

Let $V = \{0,1\}^{c'n^k}$

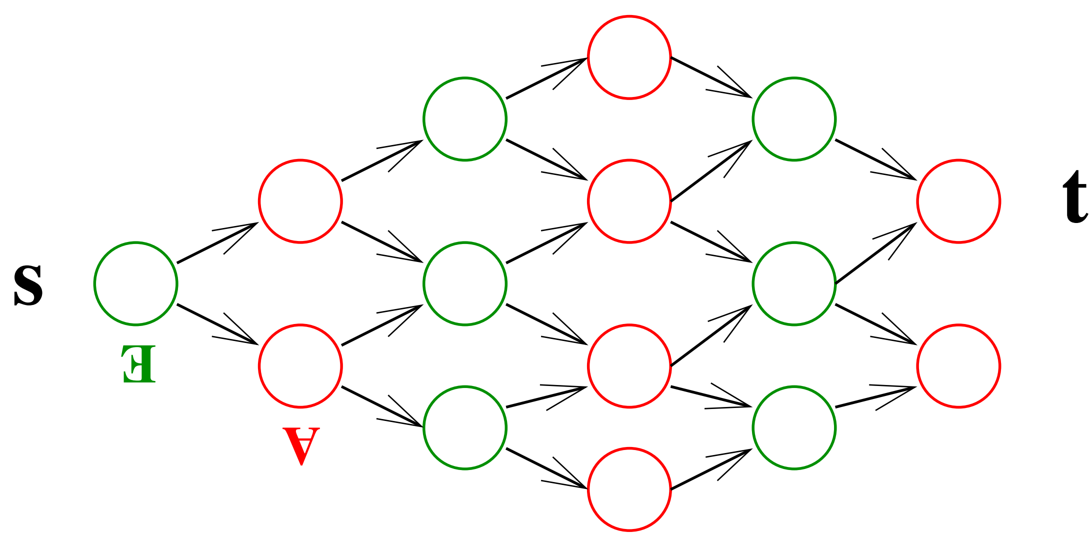
Build circuit C_w: on input $u, v \in V$, accept iff $u \underset{M}{\longrightarrow} v$.

$$M(w) = 1 \quad \Leftrightarrow \quad G(c'n^k, C_w, s, t) \in \text{SUCCINCT REACH} \qquad \square$$

The vertices of an **alternating graph**, $G = (V, A, E)$, are split into: **existential vertices** and **universal vertices**.

Def: vertex t is **reachable** from vertex s in G iff

1. $s = t$, or
2. s is existential and for some edge, $\langle s, a\rangle \in E$, t is reachable from a, or,
3. s is universal and there is an edge leaving s and for all edges, $\langle s, a\rangle \in E$, t is reachable from a.



Def: Let

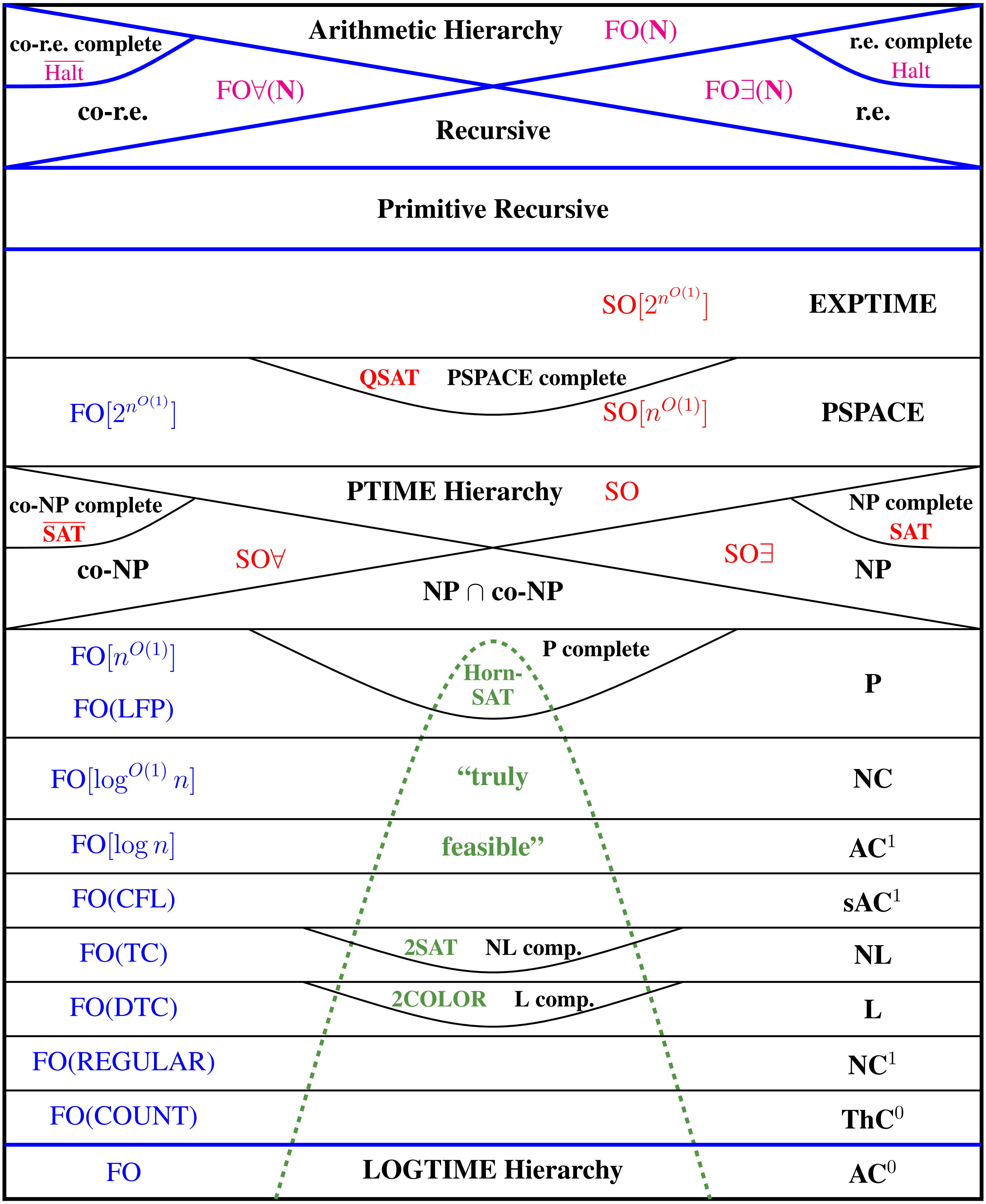
$$\text{AREACH} \;=\; \left\{ G = (V, A, E, s, t) \;\middle|\; t \text{ is reachable from } s \right\}$$

Prop: AREACH is P complete.

Proof: Think about it! (Very similar to the proof that REACH is NL complete.) $\square$

Cor: CVP and MCVP are P complete.

Proof: Easy to see that AREACH $\leq$ MCVP $\leq$ CVP. $\square$


Arithmetic Hierarchy
FO(N)
co-r.e. complete
$\overline{\text{Halt}}$
r.e. complete
Halt
FO∀(N)
FO∃(N)
co-r.e.
r.e.
Recursive
Primitive Recursive
$\mathrm{SO}[2^{n^{O(1)}}]$
EXPTIME
QSAT
PSPACE complete
$\mathrm{FO}[2^{n^{O(1)}}]$
$\mathrm{SO}[n^{O(1)}]$
PSPACE
PTIME Hierarchy
SO
co-NP complete
$\overline{\text{SAT}}$
NP complete
SAT
co-NP
SO∀
SO∃
NP
NP ∩ co-NP
$\mathrm{FO}[n^{O(1)}]$
P complete
Horn-
SAT
P
FO(LFP)
$\mathrm{FO}[\log^{O(1)} n]$
"truly
NC
$\mathrm{FO}[\log n]$
feasible"
AC^1
FO(CFL)
sAC^1
FO(TC)
2SAT
NL comp.
NL
FO(DTC)
2COLOR
L comp.
L
FO(REGULAR)
NC^1
FO(COUNT)
ThC^0
FO
LOGTIME Hierarchy
AC^0