Lecture 9: PSPACE

\[ \text{PSPACE} = \text{DSPACE}[n^{O(1)}] = \text{NSPACE}[n^{O(1)}] = \text{ATIME}[n^{O(1)}] \]

- PSPACE consists of what we could compute with a feasible amount of hardware, but with no time limit.
- PSPACE is a large and very robust complexity class.
- With polynomially many bits of memory, we can search any implicitly-defined graph of exponential size. This leads to complete problems such as reachability on exponentially-large graphs.
- We can search the game tree of any board game whose configurations are describable with polynomially-many bits and which lasts at most polynomially many moves. This leads to complete problems concerning winning strategies.
Recall \( \text{PSPACE} = \text{ATIME}[n^{O(1)}] \)

Recall QSAT, the quantified satisfiability problem.

**Proposition 9.1** QSAT is PSPACE-complete.

**Proof:** We’ve already seen that \( \text{QSAT} \in \text{ATIME}[n] \subseteq \text{PSPACE} \).

QSAT is hard for \( \text{ATIME}[n^k] \):

Let \( M \) be an \( \text{ATIME}[n^k] \) TM, \( w \) an input, \( n = |w| \)

Let \( M \) write down its \( n^k \) alternating choices, \( c_1 c_2 \ldots c_{n^k} \).

Deterministic TM \( D \) evaluates the answer, i.e., for all inputs \( w \), \( M(w) = 1 \iff \exists c_1 \forall c_2 \cdots \exists c_{n^k} (D(\overline{c}, w) = 1) \)

By Cook’s Theorem \( \exists \) reduction \( f : L(D) \leq \text{SAT} \):

\[
D(\overline{c}, w) = 1 \iff f(\overline{c}, w) \in \text{SAT}
\]

Let the new boolean variables in \( f(\overline{c}, w) \) be \( d_1 \ldots d_{t(n)} \).

\[
M(w) = 1 \iff \exists c_1 \forall c_2 \cdots \exists c_{n^k} d_1 \ldots d_{t(n)} (f(\overline{c}, w)) \in \text{QSAT}
\]

\( \square \)
Geography is a two-person game.

1. $E$ “chooses” the start vertex $v_1$.

2. $A$ chooses $v_2$, having an edge from $v_1$

3. $E$ chooses $v_3$, having an edge from $v_2$, etc.

No vertex may be chosen twice. Whoever moves last wins.
Let GEOGRAPHY be the set of positions in geography games s.t. $\exists$ has a winning strategy.

**Proposition 9.2**  GEOGRAPHY is PSPACE-complete.

**Proof:** GEOGRAPHY $\in$ PSPACE: search the polynomial-depth game tree. A polynomial-size stack suffices.

**Show:** QSAT $\leq$ GEOGRAPHY

Given formula, $\varphi$, build graph $G_\varphi$ s.t. $\exists$ chooses existential variables; $\forall$ chooses universal variables.

$$\varphi \equiv \exists a \forall b \exists c \left[ (a \lor b) \land (\overline{b} \lor c) \land (b \lor c) \right]$$

\[\square\]
**Definition 9.3** A succinct representation of a graph is \( G(n, C, s, t) = (V, E, s, t) \)
where \( C \) is a boolean circuit with \( 2n \) inputs and

\[
V = \{ w \mid w \in \{0, 1\}^n \}
\]

\[
E = \{ (w, w') \mid C(w, w') = 1 \}
\]

\[
\square
\]

\[
\text{SUCCINCT REACH} = \{ (n, C, s, t) \mid G(n, C, s, t) \in \text{REACH} \}
\]

**Proposition 9.4** \( \text{SUCCINCT REACH} \in \text{PSPACE} \)

Why?

Remember Savitch’s Thm:

\[
\text{REACH} \in \text{NSPACE}[\log n] \subseteq \text{DSPACE}[(\log n)^2]
\]

\[
\text{SUCCINCT REACH} \in \text{NSPACE}[n] \subseteq \text{DSPACE}[n^2] \subseteq \text{PSPACE}
\]

\[
\square
\]

**Proposition 9.5** \( \text{SUCCINCT REACH} \) is PSPACE-complete.

**Proof:** Let \( M \) be a DSPACE[\( n^k \)] TM, input \( w \), \( n = |w| \)

\[
M(w) = 1 \Leftrightarrow \text{CompGraph}(M, w) \in \text{REACH}
\]

\[
\text{CompGraph}(n, w) = (V, E, s, t)
\]

\[
V = \{ \text{ID} = \langle q, h, p \rangle \mid q \in \text{States}(N), h \leq n, |p| \leq cn^k \}
\]

\[
E = \{ (\text{ID}_1, \text{ID}_2) \mid \text{ID}_1(w) \xrightarrow{m} \text{ID}_2(w) \}
\]

\[
s = \text{initial ID}
\]

\[
t = \text{accepting ID}
\]

\[
\square
\]
Succinct Representation of CompGraph($n, w$):

$$
V = \{ \text{ID} = \langle q, h, p \rangle \mid q \in \text{States}(N), h \leq n, |p| \leq cn^k \} \\
E = \{ (\text{ID}_1, \text{ID}_2) \mid \text{ID}_1(w) \xrightarrow{M} \text{ID}_2(w) \}'
$$

Let $V = \{0, 1\}^c n^k$

Build circuit $C_w$: on input $u, v \in V$, accept iff $u \xrightarrow{M} v$.

$$
M(w) = 1 \iff G(c'n^k, C_w, s, t) \in \text{SUCCINCT REACH}
$$
The vertices of an alternating graph, $G = (V, A, E)$, are split into: existential vertices and universal vertices.

**Def:** vertex $t$ is **reachable** from vertex $s$ in $G$ iff

1. $s = t$, or
2. $s$ is existential and for some edge, $\langle s, a \rangle \in E$, $t$ is reachable from $a$, or,
3. $s$ is universal and there is an edge leaving $s$ and for all edges, $\langle s, a \rangle \in E$, $t$ is reachable from $a$. 

![Graph Diagram](image-url)
**Def:** Let

\[ \text{AREACH} = \{ G = (V, A, E, s, t) \mid t \text{ is reachable from } s \} \]

**Prop:** AREACH is P complete.

**Proof:** Think about it! (Very similar to the proof that REACH is NL complete.)

**Cor:** CVP and MCVP are P complete.

**Proof:** Easy to see that AREACH \( \leq \) MCVP \( \leq \) CVP.