**Thm:** REACH is complete for NL.

**Proof:** Let $A \in NL$, $A = \mathcal{L}(N)$, uses $c \log n$ bits of worktape.

Input $w$, $n = |w|$

$$w \mapsto \text{CompGraph}(N, w) = (V, E, s, t)$$

$$V = \{\text{ID} = \langle q, h, p \rangle \mid q \in \text{States}(N), h \leq n, |p| \leq c\lceil \log n \rceil\}$$

$$E = \{\text{ID}_1, \text{ID}_2 \mid \text{ID}_1(w) \xrightarrow{N} \text{ID}_2(w)\}$$

$s = \text{initial ID}$

$t = \text{accepting ID}$
Claim: \( w \in \mathcal{L}(N) \iff \text{CompGraph}(N, w) \in \text{REACH} \)

\[
\begin{array}{c}
\text{s} \\
\text{v} \\
\text{t}
\end{array}
\]

Cor: \( \text{NL} \subseteq \text{P} \)

Proof: \( \text{REACH} \in \text{P} \)

\( \text{P} \) is closed under (logspace) reductions.

i.e., \( (B \in \text{P} \land A \leq B) \implies A \in \text{P} \)

\( \square \)
NSPACE vs. DSPACE

**Proposition 6.1** \( \text{NSPACE}[s(n)] \subseteq \text{NTIME}[2^{O(s(n))}] \subseteq \text{DSpace}[2^{O(s(n))}] \)

We can do much better!

**Theorem 6.2** [Savich] \( \text{REACH} \in \text{DSpace}[(\log n)^2] \)

\[
\begin{align*}
\text{Proof:} & \\
G \in \text{REACH} & \quad \iff \quad G \models \text{PATH}(s, t, n) \\
\text{PATH}(x, y, 1) & \equiv x = y \lor E(x, y) \\
\text{PATH}(x, y, 2d) & \equiv \exists z (\text{PATH}(x, z, d) \land \text{PATH}(z, y, d))
\end{align*}
\]

\( S_n(d) \) = space to check paths of dist. \( d \) in \( n \)-nodegraphs

\[
S_n(n) = \log n + S_n(n/2) = O((\log n)^2)
\]
Savitch’s Thm: For \( s(n) \geq \log n \),

\[
\text{DSPACE}[s(n)] \subseteq \text{NSPACE}[s(n)] \subseteq \text{DSPACE}[(s(n))^2]
\]

Proof: Let \( A \in \text{NSPACE}[s(n)] \); \( A = L(N) \)

\[
w \in A \iff \text{CompGraph}(N, w) \in \text{REACH}
\]

\[
|w| = n; \quad |\text{CompGraph}(N, w)| = 2^{O(s(n))}
\]

Testing if \( \text{CompGraph}(N, w) \in \text{REACH} \) takes space,

\[
(\log(|\text{CompGraph}(N, w)|))^2 = (\log(2^{O(s(n))}))^2 = O((s(n))^2)
\]

From \( w \) build \( \text{CompGraph}(N, w) \) in \( \text{DSPACE}[s(n)] \). \( \square \)
Theorem 6.3 \( \text{REACH} \in \text{NL} \)

**Proof:** Fix \( G \), let \( N_d = \{ v \mid \text{distance}(s, v) \leq d \} \)

**Claim:** The following problems are in NL:

1. \( \text{DIST}(x, d) \): distance\((s, x) \leq d \)
2. \( \text{NDIST}(x, d; m) \): if \( m = N_d \) then \( \neg \text{DIST}(x, d) \)

**Proof:**

1. Guess the path of length \( \leq d \) from \( s \) to \( x \).
2. Guess \( m \) vertices, \( v \neq x \), with \( \text{DIST}(v, d) \).

\[
c := 0; \\
\text{for } v := 1 \text{ to } n \text{ do } \{ \text{// nondeterministically} \} \\
\quad ( \text{DIST}(v, d) \&\& v \neq x; c++; || ) \\
\quad ( \text{no-op} ) \\
\} \\
\text{if } (c == m) \text{ then ACCEPT}
\]
Claim: We can compute $N_d$ in NL.

Proof: By induction on $d$.

Base case: $N_0 = 1$

Inductive step: Suppose we have $N_d$.

1. $c := 0$;
2. for $v := 1$ to $n$ do { // nondeterministically
3.  ( DIST($v, d+1$); $c + +$ ) ||
4.  ( $\forall z \left( \text{NDIST}(z, d; N_d) \lor (z \neq v \land \neg E(z, v)) \right)$
5. }
6. $N_{d+1} := c$

$G \in \text{REACH} \iff N\text{DIST}(t, n; N_n)$

Theorem 6.4 [Immerman-Szelepcsényi] If $s(n) \geq \log n$, Then, NSPACE[$s(n)$] = co-NSPACE[$s(n)$]

Proof: Let $A \in \text{NSPACE}[s(n)]; \ A = \mathcal{L}(N)$

\[ w \in A \iff \text{CompGraph}(N, w) \in \text{REACH} \]

\[ |w| = n; \quad |\text{CompGraph}(N, w)| = 2^{O(s(n))} \]

Testing if $\text{CompGraph}(N, w) \in \text{REACH}$ takes space,

\[
\log(|\text{CompGraph}(N, w)|) = \log(2^{O(s(n))}) = O(s(n))
\]