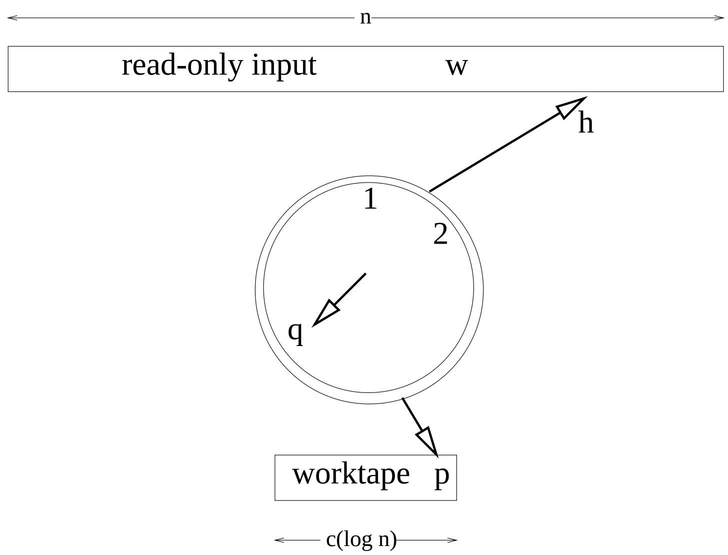
Thm: REACH is complete for NL.

Proof: Let $A \in \text{NL}$, $A = \mathcal{L}(N)$, uses $c \log n$ bits of worktape.

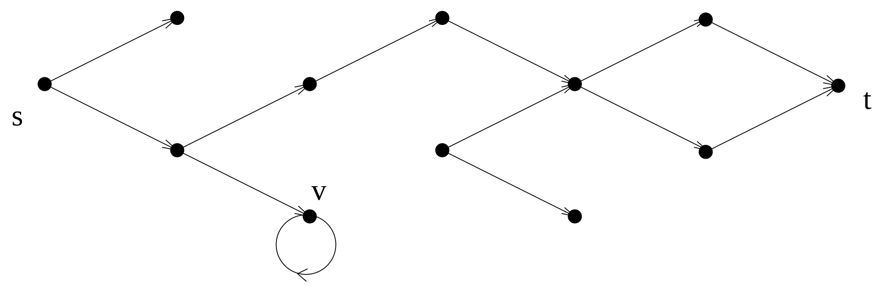
Input w, $\quad n = |w|$

$$w \;\mapsto\; \text{CompGraph}(N, w) \quad = \quad (V, E, s, t)$$

$$\begin{aligned}
V &= \left\{ \text{ID} = \langle q, h, p \rangle \;\middle|\; q \in \text{States}(N), h \le n, |p| \le c\lceil \log n \rceil \right\} \\
E &= \left\{ (\text{ID}_1, \text{ID}_2) \;\middle|\; \text{ID}_1(w) \underset{N}{\longrightarrow} \text{ID}_2(w) \right\} \\
s &= \text{initial ID} \\
t &= \text{accepting ID}
\end{aligned}$$



Claim: $w \in \mathcal{L}(N) \;\Leftrightarrow\;$ CompGraph$(N, w) \in$ REACH



$\square$

Cor: NL $\subseteq$ P

Proof: REACH $\in$ P

P is closed under (logspace) reductions.

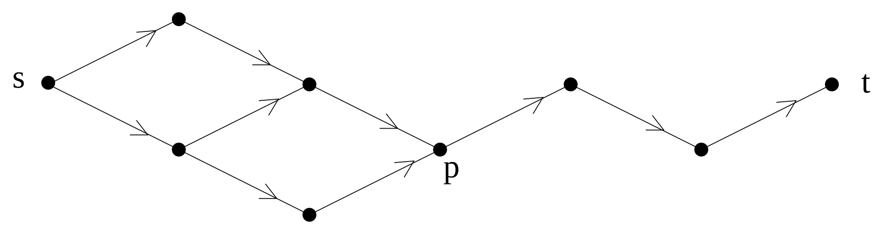
i.e., $(B \in \mathrm{P} \quad \wedge \quad A \leq B) \quad \Rightarrow \quad A \in \mathrm{P}$ $\square$

NSPACE vs. DSPACE

Proposition 6.1 NSPACE$[s(n)] \subseteq$ NTIME$[2^{O(s(n))}] \subseteq$ DSPACE$[2^{O(s(n))}]$

We can do much better!

Theorem 6.2 *[*Savich] REACH $\in$ DSPACE$[(\log n)^2]$



Proof:

$$\begin{aligned} G \in \text{REACH} &\Leftrightarrow G \models \text{PATH}(s,t,n) \\ \text{PATH}(x,y,1) &\equiv x = y \lor E(x,y) \\ \text{PATH}(x,y,2d) &\equiv \exists z\,(\text{PATH}(x,z,d) \land \text{PATH}(z,y,d)) \end{aligned}$$

$S_n(d)$ = space to check paths of dist. d in n-nodegraphs

$$\begin{aligned} S_n(n) &= \log n + S_n(n/2) \\ &= O((\log n)^2) \end{aligned}$$

$\square$

Savitch's Thm: For $s(n) \geq \log n$,

$$\text{DSPACE}[s(n)] \;\subseteq\; \text{NSPACE}[s(n)] \;\subseteq\; \text{DSPACE}[(s(n))^2]$$

Proof: Let $A \in \text{NSPACE}[s(n)]$; $\quad A = \mathcal{L}(N)$

$$w \in A \qquad \Leftrightarrow \qquad \text{CompGraph}(N, w) \in \text{REACH}$$

$$|w| = n; \qquad |\text{CompGraph}(N, w)| = 2^{O(s(n))}$$

Testing if $\text{CompGraph}(N, w) \in \text{REACH}$ takes space,

$$\begin{aligned}(\log(|\text{CompGraph}(N, w)|))^2 &= (\log(2^{O(s(n))}))^2 \\ &= O((s(n))^2)\end{aligned}$$

From w build $\text{CompGraph}(N, w)$ in DSPACE[s(n)]. $\square$

Theorem 6.3 $\overline{\text{REACH}} \in \text{NL}$

Proof: Fix G, let $N_d = \left|\left\{v \mid \text{distance}(s,v) \le d\right\}\right|$

Claim: The following problems are in NL:

1. $\text{DIST}(x,d)$: $\text{distance}(s,x) \le d$
2. $\text{NDIST}(x,d;m)$: if $m = N_d$ then $\neg\text{DIST}(x,d)$

Proof:

1. Guess the path of length $\le d$ from s to x.
2. Guess m vertices, $v \ne x$, with $\text{DIST}(v,d)$.

```
c := 0;
for v := 1 to n do {    // nondeterministically
    ( DIST(v, d) && v ≠ x; c++ )    ||
    ( no-op )
}
if (c == m) then ACCEPT
```

$\square$

Claim: We can compute N_d in NL.

Proof: By induction on d.

Base case: $N_0 = 1$

Inductive step: Suppose we have N_d.

1. $c := 0$;
2. **for** $v := 1$ to n **do** { // nondeterministically
3. ($\text{DIST}(v, d+1); c++$) $||$
4. $(\forall z\,(\text{NDIST}(z, d; N_d)\ \vee\ (z \neq v \wedge \neg E(z, v))))$
5. }
6. $N_{d+1} := c$

$\square$

$$G \in \overline{\text{REACH}} \quad \Leftrightarrow \quad \text{NDIST}(t, n; N_n)$$ $\square$

Theorem 6.4 [Immerman-Szelepcsényi] *If* $s(n) \geq \log n$, *Then,* $\text{NSPACE}[s(n)] \;=\; co\text{-NSPACE}[s(n)]$

Proof: Let $A \in \text{NSPACE}[s(n)]$; $A = \mathcal{L}(N)$

$$w \in A \qquad \Leftrightarrow \qquad \text{CompGraph}(N, w) \in \text{REACH}$$

$$|w| = n; \qquad |\text{CompGraph}(N, w)| = 2^{O(s(n))}$$

Testing if $\text{CompGraph}(N, w) \in \overline{\text{REACH}}$ takes space,

$$\begin{aligned} \log(|\text{CompGraph}(N, w)|) &= \log(2^{O(s(n))}) \\ &= O(s(n)) \end{aligned}$$

$\square$