Pumping Lemma for Regular Sets:
Let $D = (Q, \Sigma, \delta, q_0, F)$ be a DFA.

Let $n = |Q|$.

Let $w \in \mathcal{L}(D)$ s.t. $|w| \geq n$.

Then $\exists x, y, z \in \Sigma^*$ s.t. the following all hold:

1. $xyz = w$
2. $|xy| \leq n$
3. $|y| > 0$, and
4. $\forall k \geq 0 \ (xy^kz \in \mathcal{L}(D))$
**proof:** Let $w \in \mathcal{L}(D)$, $|w| \geq n$, $w = w_1, w_2, \ldots, w_n \cdot u$

\[
\begin{array}{cccccccc}
w &=& w_1 & w_2 & w_3 & \cdots & w_n & u \\
q_0 & q_1 & q_2 & q_3 & \cdots & q_{n-1} & q_n & q_f
\end{array}
\]

By **Pigeon-Hole Principle** $\exists i < j (q_i = q_j)$

\[
w = \underbrace{\begin{array}{c}w_1 \ldots w_i \end{array}}_{x} \underbrace{\begin{array}{c}w_{i+1} \ldots w_j \end{array}}_{y} \underbrace{\begin{array}{c}w_{j+1} \ldots w_n u \end{array}}_{z} \]

$q_i = \delta^*(q_i, y)$ $|y| \geq 1$ $\delta^*(q_i, z) = q_f \in F$

Thus, $x y^k z \in \mathcal{L}(D)$ for $k = 0, 1, 2, \ldots$ □
Pumping Lemma for Regular Sets:

Let \( D = (Q, \Sigma, \delta, q_0, F) \) be a DFA.

Let \( n = |Q| \).

Let \( w \in \mathcal{L}(D) \) s.t. \( |w| \geq n \).

Then \( \exists x, y, z \in \Sigma^* \) s.t. the following all hold:

- \( xyz = w \)
- \( |xy| \leq n \)
- \( |y| > 0 \), and
- \( \forall k \geq 0 \ (xy^kz \in \mathcal{L}(D)) \)

Easiest tool to prove languages not regular
Prop: $E = \{ a^r b^r \mid r \in \mathbb{N} \}$ is not regular.

proof: Suppose $E$ is accepted by DFA $D$ with $n$ states.

you choose string: $w \in E = \mathcal{L}(D)$ to get contradiction

Let $w = a^n b^n$

By pumping lemma, $w = a^n b^n = xyz$ s.t.

- $|xy| \leq n$
- $|y| > 0$, and
- $\forall k \in \mathbb{N} (xy^k z \in E)$

Since $0 < |xy| \leq n$, $y = a^i, 0 < i \leq n$

Thus $xy^0 z = a^{n-i} b^n \in E$.

$a^{n-i} b^n \notin E$.

Therefore $E$ is not regular. \( \Box \)
**Prop:** \( M = \{w \in \{a, b\}^* \mid \#_a(w) > \#_b(w)\} \) not regular.

**proof:** Suppose \( M \) is accepted by DFA \( D \) with \( n \) states.

You choose string: \( w \in M = \mathcal{L}(D) \) to get contradiction

Let \( w = a^{n+1}b^n \)

By pumping lemma, \( w = a^{n+1}b^n = xyz \) s.t.

- \(|xy| \leq n\)
- \(|y| > 0\), and
- \( \forall k \in \mathbb{N} (xy^kz \in M) \)

Since \( 0 < |xy| \leq n \), \( y = a^i, 0 < i \leq n \)

Thus \( xy^0z = a^{n+1-i}b^n \in M \).

\( a^{n+1-i}b^n \not\in M \).

Therefore \( M \) is not regular. \( \square \)
Prop: $P = \{ w \in \{a, b\}^* \mid |w| \text{ is prime} \}$ is not regular.

proof: Suppose $P$ is accepted by DFA $D$ with $n$ states.

you choose string: $w \in P = \mathcal{L}(D)$ to get contradiction

Let $w = a^p$ where $p \geq n$ is prime

By pumping lemma, $w = a^p = xyz$ s.t.

- $|xy| \leq n$
- $|y| > 0$, and
- $\forall k \in \mathbb{N}(xy^kz \in P)$

$y = a^i, 0 < i \leq n$

Thus $xy^{p+1}z = xyz^p = a^p a^{p \cdot i} = a^{p(i+1)} \in P$.

But $p(i + 1)$ is not prime, so $xy^{p+1}z \notin P$.

Therefore $P$ is not regular.