Theorem 7.1 *Cook's Thm:* SAT *is* NP-*complete.*

Proof:

Let $B \in$ NP. By Fagin's theorem,

$$B \;=\; \{\mathcal{A} \mid \mathcal{A} \models \Phi\}; \qquad \Phi \;\equiv\; \exists C_0^{2k} \cdots C_{g-1}^{2k} \Delta^k \, \forall x_1 \cdots x_t \, (\alpha(\overline{x}))$$

with α quantifier-free and CNF,

$$\alpha(\bar{x}) \quad = \quad \bigwedge_{i=1}^{r} \bigvee_{j=1}^{s} \lambda_{i,j}(\bar{x})$$

where each $\lambda_{i,j}$ is a literal.

$\mathcal{A}$ arbitrary, $n = \|\mathcal{A}\|$, Define boolean formula $\varphi_{\mathcal{A}}$:

boolean variables: $C_i(e_1, \ldots, e_{2k}),\ \Delta(e_1, \ldots, e_k)$

$i = 1, \ldots, g,\ e_1, \ldots, e_{2k} \in |\mathcal{A}|$

literals: $\lambda_{i,j}(\bar{e})$, $i = 1, \ldots, r$, $j = 1, \ldots, s$, $\bar{e} \in |\mathcal{A}|^t$

$\lambda'_{i,j}(\bar{e})$ is $\lambda_{i,j}(\bar{e})$ with $R(\bar{e})$, replaced by $\top$ or $\bot$ according as $\mathcal{A} \models R(\bar{e})$; $C_i(\bar{e})$, and $\Delta(\bar{e})$ are just boolean variables.

$$\begin{aligned}
\Phi \;&\equiv\; \exists C_0^{2k} \cdots C_{g-1}^{2k} \Delta^k \, \forall x_1 \cdots x_t \bigwedge_{i=1}^{r} \bigvee_{j=1}^{s} \lambda_{i,j}(\bar{x}) \\
\varphi(\mathcal{A}) \;&\equiv\; \bigwedge_{e_1, \ldots, e_t \in |\mathcal{A}|} \bigwedge_{i=1}^{r} \bigvee_{j=1}^{s} \lambda'_{i,j}(\bar{e})
\end{aligned}$$

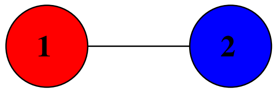
$$\mathcal{A} \in B \qquad \Leftrightarrow \qquad \mathcal{A} \models \Phi \qquad \Leftrightarrow \qquad \varphi(\mathcal{A}) \in \text{SAT}$$

$\square$

Example: Fagin's Theorem implies Cook's Theorem

$$\begin{aligned}\Phi_{\text{2-color}} \;\equiv\; & \exists R^1\,\exists B^1\,\forall x,y\,\big[(R(x)\vee B(x))\\ & \quad\wedge\quad (\neg E(x,y)\vee\neg R(x)\vee\neg R(y))\\ & \quad\wedge\quad (\neg E(x,y)\vee\neg B(x)\vee\neg B(y))\big]\end{aligned}$$

G



boolean variables: r_1, r_2, b_1, b_2

$$\begin{aligned}\varphi_G \;\equiv\; & (r_1\vee b_1)\wedge(\top\vee\overline{r_1}\vee\overline{r_1})\wedge(\top\vee\overline{b_1}\vee\overline{b_1})\\ & (r_1\vee b_1)\wedge(\bot\vee\overline{r_1}\vee\overline{r_2})\wedge(\bot\vee\overline{b_1}\vee\overline{b_2})\\ & (r_2\vee b_2)\wedge(\bot\vee\overline{r_2}\vee\overline{r_1})\wedge(\bot\vee\overline{b_2}\vee\overline{b_1})\\ & (r_2\vee b_2)\wedge(\top\vee\overline{r_2}\vee\overline{r_2})\wedge(\top\vee\overline{b_2}\vee\overline{b_2})\end{aligned}$$

Simplifies to: $(r_1\vee b_1)\wedge(r_2\vee b_2)\wedge(\overline{r_1}\vee\overline{r_2})\wedge(\overline{b_1}\vee\overline{b_2})$

$$\text{3-SAT} \quad = \quad \left\{ \varphi \in \text{CNF-SAT} \;\middle|\; \varphi \text{ has} \le 3 \text{ literals per clause} \right\}$$

Prop: 3-SAT is NP-complete.

Proof: Show SAT $\le$ 3-SAT.

Example: $C \quad \equiv \quad (\ell_1 \vee \ell_2 \vee \cdots \vee \ell_7)$

$$\begin{aligned} C' \quad \equiv \quad & (\ell_1 \vee \ell_2 \vee d_1) \wedge (\overline{d_1} \vee \ell_3 \vee d_2) \wedge (\overline{d_2} \vee \ell_4 \vee d_3) \wedge \\ & (\overline{d_3} \vee \ell_5 \vee d_4) \wedge (\overline{d_4} \vee \ell_6 \vee \ell_7) \end{aligned}$$

Claim 7.2 $C \in \text{SAT} \quad \Leftrightarrow \quad C' \in \text{3-SAT}$

Do this construction for each clause independently. $\square$

Working assumption: 3-SAT requires $2^{\epsilon n}$ time.

Proposition 7.3 3-COLOR *is* NP-*complete (NPC).*

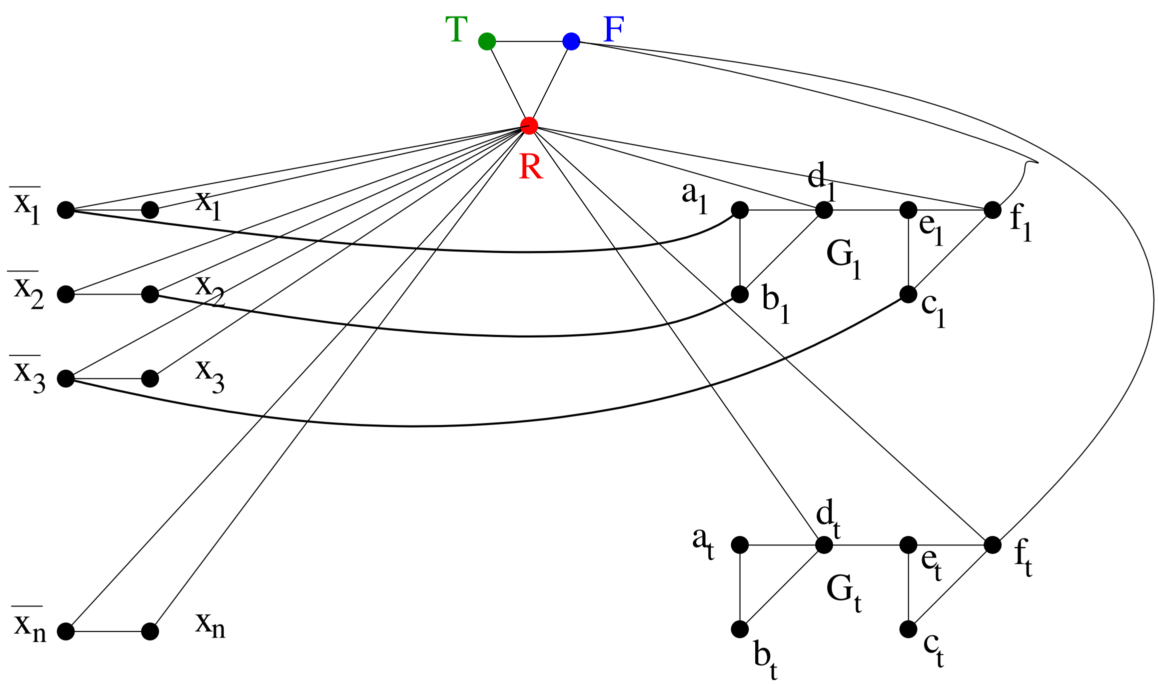
Proof: Saw previously that 3-COLOR $\in$ SO$\exists$ = NP.

Show 3-SAT $\leq$ 3-COLOR:

$$\varphi \equiv C_1 \wedge C_2 \wedge \cdots \wedge C_t \quad \in \quad \text{3-CNF}; \qquad \mathbf{var}(\varphi) = \left\{x_1, x_2, \ldots, x_n\right\}$$

Must build graph $G(\varphi)$ s.t.

$$\varphi \in \text{3-SAT} \qquad \Leftrightarrow \qquad G(\varphi) \in \text{3-COLOR}$$



G_1 encodes clause $C_1 = (\overline{x_1} \vee x_2 \vee \overline{x_3})$

Claim 7.4 *Triangle a_1, b_1, d_1 serves as an "or"-gate: d_1* **may** *be colored "true" iff at least one of its inputs $\overline{x_1}, x_2$ is colored "true".*

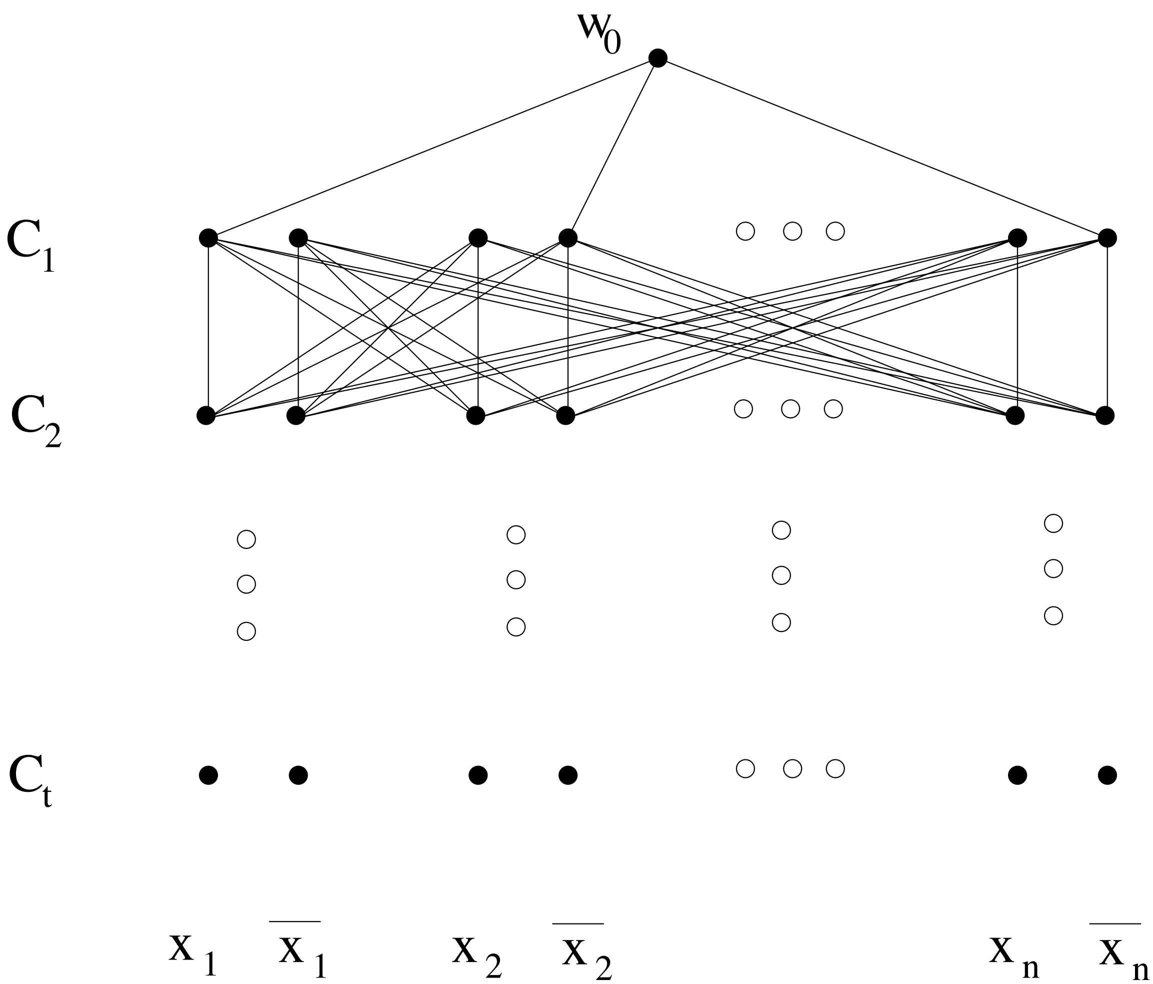
A three coloring of the literals can be extended to color G_i iff the corresponding truth assignment makes C_i true.
□

Proposition 7.5 CLIQUE *is* NP-*complete.*

Proof: Show SAT $\leq$ CLIQUE.

$$\varphi \;\equiv\; C_1 \;\wedge\; C_2 \;\wedge\; \cdots \;\wedge\; C_t \in \text{3-CNF}, \quad C_1 = (x_1 \vee \overline{x_2} \vee \overline{x_n}), \quad \textbf{var}(\,\varphi) \;=\; \Big\{x_1, x_2, \ldots, x_n\Big\}$$

Must build graph $g(\,\varphi)$ s.t. $\quad \varphi \in \text{SAT} \quad \Leftrightarrow \quad g(\,\varphi) \in \text{CLIQUE}$



$$g(\,\varphi) \;=\; (V^{g(\,\varphi)}, E^{g(\,\varphi)}, k^{g(\,\varphi)}), \quad k^{g(\,\varphi)} = t+1$$

$$V^{g(\,\varphi)} \;=\; (C \times L) \cup \{w_0\}, \quad L = \Big\{x_1, \ldots, x_n, \overline{x}_1, \ldots, \overline{x}_n\Big\}; \quad C = \Big\{c_1, \ldots, c_t\Big\}$$

$$E^{g(\,\varphi)} \;=\; \Big\{(\langle c_1, \ell_1\rangle, \langle c_2, \ell_2\rangle) \;\Big|\; c_1 \neq c_2 \text{ and } \overline{\ell}_1 \neq \ell_2\Big\} \;\cup\; \Big\{(w_0, \langle c, \ell\rangle), (\langle c, \ell\rangle, w_0) \;\Big|\; \ell \text{ occurs in } c\Big\}$$

$\square$

$$\textbf{Subset Sum} \quad = \quad \left\{ m_1, \ldots, m_r, T \in \mathbf{N} \;\middle|\; \exists S \subseteq \{1, \ldots, r\} \left(\sum_{i \in S} m_i = T \right) \right\}$$

Proposition 7.6 **Subset Sum** *is* NP-*Complete.*

Show 3-SAT $\leq$ Subset Sum.

$$\varphi \equiv C_1 \wedge C_2 \wedge \cdots \wedge C_t \quad \in \quad \text{3-CNF}; \qquad \textbf{var}(\varphi) = \left\{ x_1, x_2, \ldots, x_n \right\}$$

Build $f \in$ FL such that for all φ, $\quad \varphi \in$ 3-SAT $\quad \Leftrightarrow \quad f(\varphi) \in$ Subset Sum

	x_1	x_2	$\cdots$	x_n	C_1	C_2	$\cdots$	C_t	
T	1	1	$\cdots$	1	3	3	$\cdots$	3	
x_1	1	0	$\cdots$	0	1	0	$\cdots$	1	$C_1 = (x_1 \vee \overline{x_2} \vee x_3)$
$\overline{x_1}$	1	0	$\cdots$	0	0	1	$\cdots$	0	
x_2	0	1	$\cdots$	0	0	1	$\cdots$	1	$C_2 = (\overline{x_1} \vee x_2 \vee x_n)$
$\overline{x_2}$	0	1	$\cdots$	0	1	0	$\cdots$	0	
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$C_t = (x_1 \vee x_2 \vee \overline{x_n})$
x_n	0	0	$\cdots$	1	0	1	$\cdots$	0	
$\overline{x_n}$	0	0	$\cdots$	1	0	0	$\cdots$	1	
a_1	0	0	$\cdots$	0	1	0	$\cdots$	0	
b_1	0	0	$\cdots$	0	1	0	$\cdots$	0	
a_2	0	0	$\cdots$	0	0	1	$\cdots$	0	
b_2	0	0	$\cdots$	0	0	1	$\cdots$	0	
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	
a_t	0	0	$\cdots$	0	0	0	$\cdots$	1	
b_t	0	0	$\cdots$	0	0	0	$\cdots$	1	

Knapsack: n objects; W = max weight I can carry in my knapsack.

object	o_1	o_2	$\cdots$	o_n	
weight	w_1	w_2	$\cdots$	w_n	≥ 0
value	v_1	v_2	$\cdots$	v_n	

Optimization Problem: choose $S \subseteq \{1, \ldots, n\}$ to maximize $\sum_{i \in S} v_i$ such that $\sum_{i \in S} w_i \leq W$

Decision Problem: Given $\bar{w}, \bar{v}, W, V$, can I get total value $\geq V$ while total weight is $\leq W$?

Proposition 7.7 *Knapsack is* NP-*Complete.*

Proof:

Let $I = \langle m_1, \ldots m_n, T \rangle$ be a Subset Sum instance.

Problem: $\exists ? S \subseteq \{1, \ldots, n\} \left(\sum_{i \in S} m_i = T \right)$

$f(I) = \langle m_1, \ldots m_n, m_1, \ldots, m_n, T, T \rangle$ is a Knapsack instance.

Claim: $I \in$ Subset Sum $\Leftrightarrow$ $f(I) \in$ Knapsack

$$\exists S \subseteq \{1, \ldots, n\} \left(\sum_{i \in S} m_i = T \right) \qquad \Leftrightarrow$$

$$\exists S \subseteq \{1, \ldots, n\} \left(\sum_{i \in S} m_i \geq T \quad \wedge \quad \sum_{i \in S} m_i \leq T \right)$$

$\square$

Fact 7.8 *Even though Knapsack is* NP-*Complete there is an efficient dynamic programming algorithm that can closely approximate the maximum possible V.*

Knapsack and Subset Sum are NP complete decision problems when we have to get the answer exactly right on all polynomially many digits. We can efficiently get it right to 10 (in fact to $\log n$) digits of accuracy.