Lecture 17: $\text{NC}^1$ and Barrington’s Theorem
Theorem 17.1  The set of problems accepted by uniform (polynomial size) branching programs is $\text{DSPACE}[\log n]$.

\[ \text{BranchingPrograms} = \text{L} \]

Proof:

BranchingPrograms $\subseteq$ L: just keep track of where you are!

L $\subseteq$ BranchingPrograms:

Let $M$ be a $\text{DSPACE}[\log n]$ Turing machine.

The computation graph of $M$ on some variable input $x_1 \cdots x_n$ is a branching program! $\square$
Proposition 17.2 *The set of problems accepted by uniform, bounded-width branching programs is contained in NC^1.*

**Proof:** This is similar to the proof that REACH ∈ sAC^1. However, instead of \( n \) choices to guess the middle point, there are only a bounded number of choices. \( \square \)
Bounded Width Branching Programs look very much like finite automata.

\[
\text{MAJ} = \{ w \in \{0, 1\}^* \mid \text{w contains more than } |w|/2 \text{ “1”s} \}\n\]

Natural Conjecture:

\[
\text{MAJ} \not\in \text{Bounded Width BPs}
\]
$S_5$ is the permutation group on 5 objects.

$$\alpha = (12345), \quad \beta = (13542) \in S_5$$

$$[\alpha, \beta] = \alpha \beta \alpha^{-1} \beta^{-1}$$

$$= (12345)(13542)(54321)(24531)$$

$$= (13254)$$
Definition 17.3 A width 5 Branching Program, $B$, 5-cycle recognizes $S$ iff for some 5-cycle $\sigma$,

- For $x \in S$, $B(x) = \sigma$
- For $x \notin S$, $B(x) = e$

Lemma 17.4 Let $S_i = \{x \in \{0, 1\}^n \mid x_i = 1\}$.
$S_i$ can be 5-cycle recognized.

Lemma 17.5 If $S$ is 5-cycle recognized, then so is $\overline{S}$
Lemma 17.6 If \( S \) is 5-cycle recognized using 5-cycle \( \sigma \), then \( S \) can be 5-cycle recognized using 5-cycle \( \tau \).

Proof: Every two 5-cycles are conjugates, i.e.,
\[
(\exists \theta \in S_5)( \tau = \theta^{-1} \sigma \theta )
\]

Lemma 17.7 If \( S \) and \( T \) can be 5-cycle recognized by branching programs \( B \) and \( C \), then \( S \cap T \) can be 5-cycle recognized by a branching program of size \( 2(|B| + |C|) \).

Proof:
\[
B \quad C \quad B^{-1} \quad C^{-1}
\]
Theorem 17.8 (Barrington’s Theorem)

\[ \text{Bounded Width Branching Programs} = \text{NC}^1 \]

Proof:

Given an NC\(^1\) circuit, simulate it using the above lemmas.

We multiply the size of the branching programs by 4 as we go up one level.

Total size is \(4^{O(\log n)} = n^{O(1)}\)