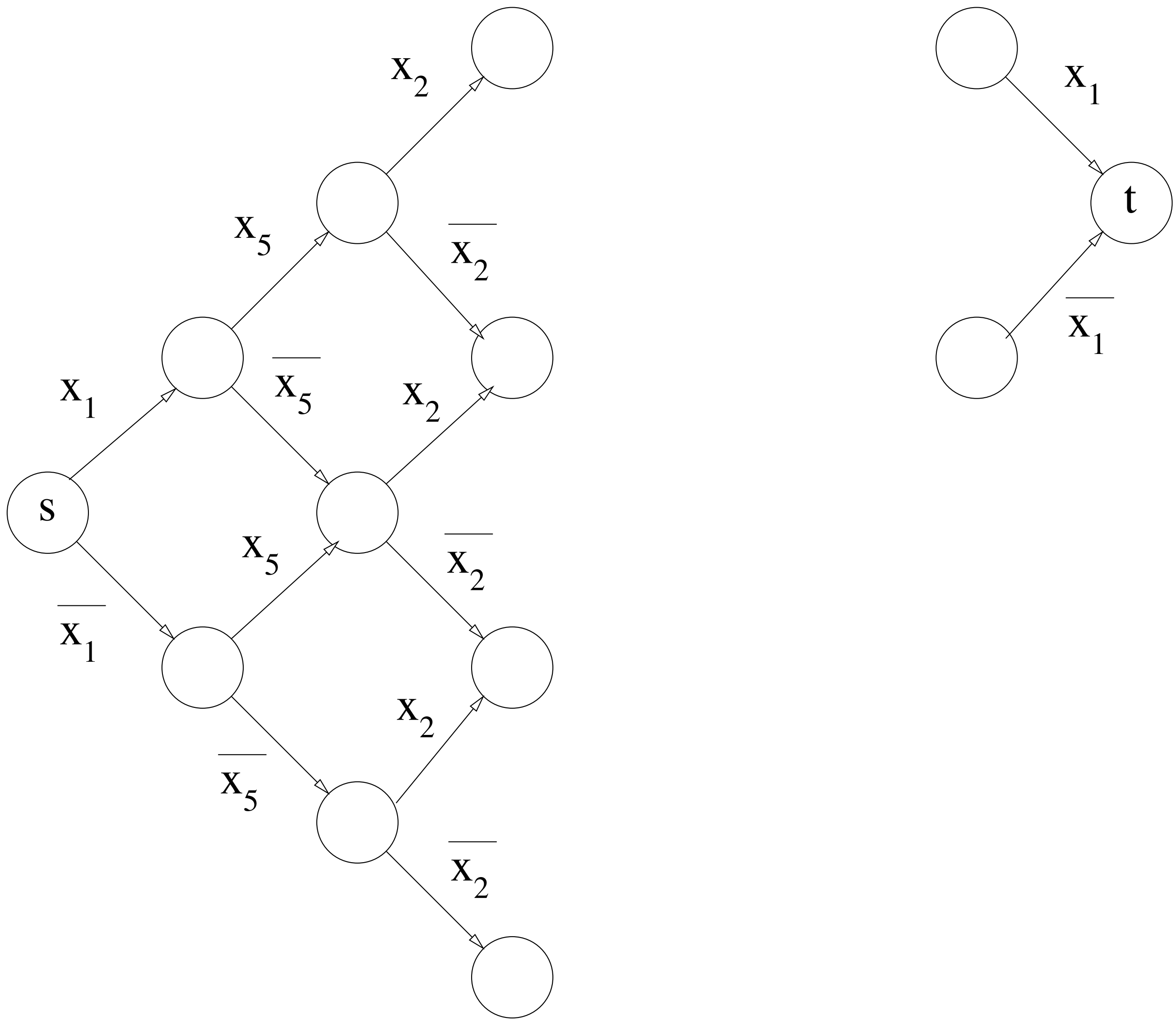
Lecture 17: NC^1 and Barrington's Theorem



Theorem 17.1 *The set of problems accepted by uniform (polynomial size) branching programs is* DSPACE$[\log n]$.

$$\textit{BranchingPrograms} \quad = \quad \mathrm{L}$$

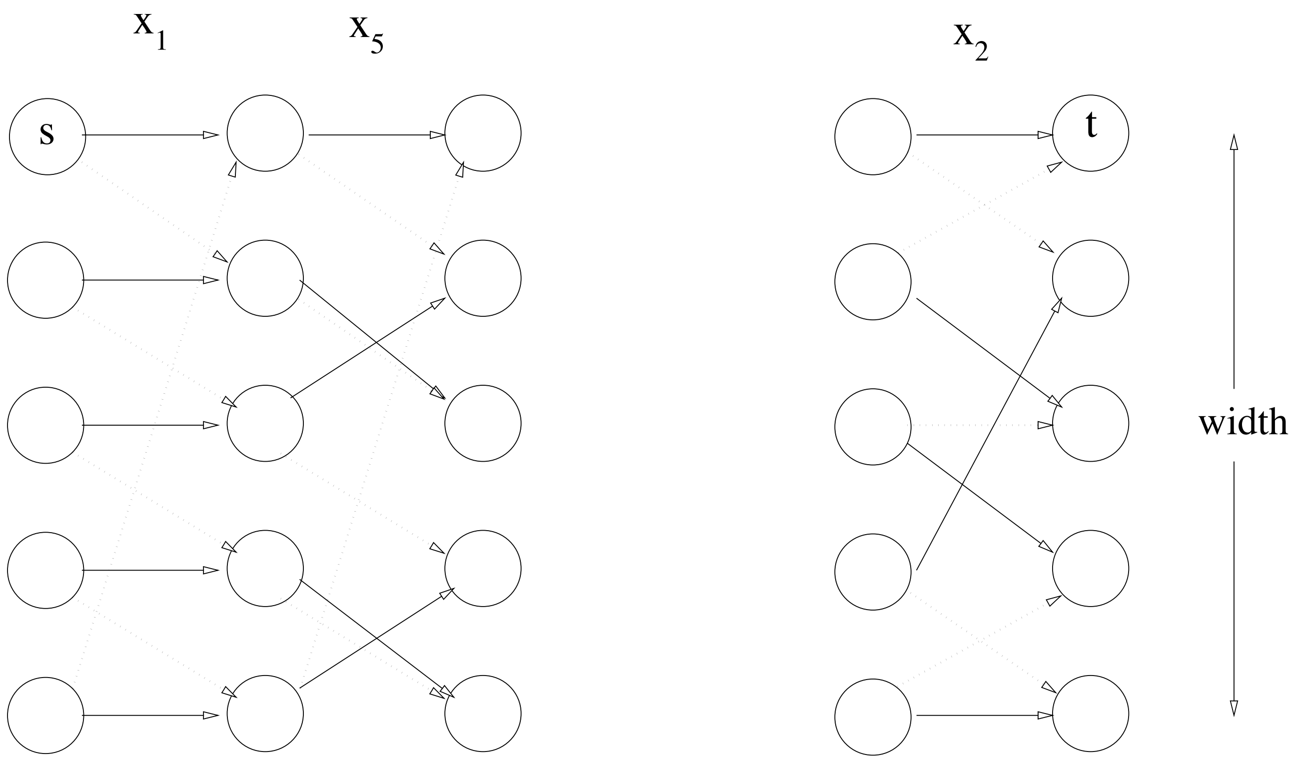
Proof:

BranchingPrograms $\subseteq$ L: just keep track of where you are!

L $\subseteq$ BranchingPrograms:

Let M be a DSPACE$[\log n]$ Turing machine.

The computation graph of M on some variable input $x_1 \cdots x_n$ is a branching program! $\square$



Proposition 17.2 *The set of problems accepted by uniform, bounded-width branching programs is contained in* NC^1.

Proof: This is similar to the proof that REACH $\in$ sAC^1. However, instead of n choices to guess the middle point, there are only a bounded number of choices. $\square$

Bounded Width Branching Programs look very much like finite automata.

$$\text{MAJ} \;=\; \left\{w \in \{0,1\}^\star \;\middle|\; w \text{ contains more than } |w|/2 \text{ “1”s}\right\}$$

Natural Conjecture:

$$\text{MAJ} \notin \text{Bounded Width BPs}$$

S_5 is the permutation group on 5 objects.

$$\alpha = (12345), \quad \beta = (13542) \in S_5$$

$$\begin{aligned}
[\alpha, \beta] &= \alpha\beta\alpha^{-1}\beta^{-1} \\
&= (12345)(13542)(54321)(24531) \\
&= (13254)
\end{aligned}$$

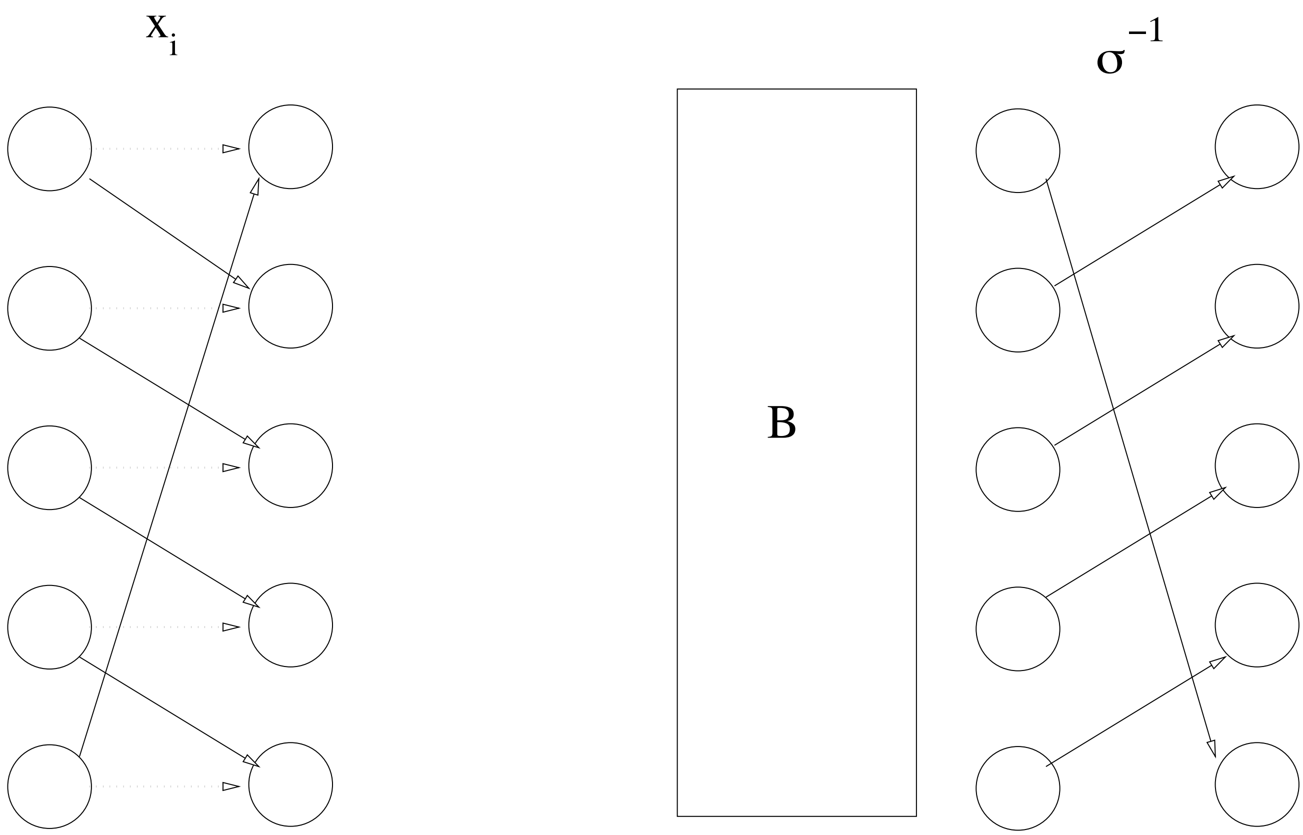
Definition 17.3 A width 5 Branching Program, B, *5-cycle recognizes* S iff for some 5-cycle σ,

- For $x \in S$, $B(x) = \sigma$
- For $x \notin S$, $B(x) = e$

$\square$

Lemma 17.4 *Let* $S_i = \left\{x \in \{0,1\}^n \;\middle|\; x_i = 1\right\}$.
S_i *can be 5-cycle recognized.*

Lemma 17.5 *If* S *is 5-cycle recognized, then so is* $\overline{S}$



Lemma 17.6 *If S is 5-cycle recognized using 5-cycle σ, then S can be 5-cycle recognized using 5-cyle τ.*

Proof: Every two 5-cycles are conjugates, i.e.,

$$(\exists \theta \in S_5)(\ \tau \ = \ \theta^{-1}\sigma\theta\ )$$

$\square$

Lemma 17.7 *If S and T can be 5-cycle recognized by branching programs B and C, then $S \cap T$ can be 5-cycle recognized by a branching program of size $2(|B| + |C|)$*

Proof:

$$B \quad C \quad B^{-1} \quad C^{-1}$$

$\square$

Theorem 17.8 (Barrington's Theorem)

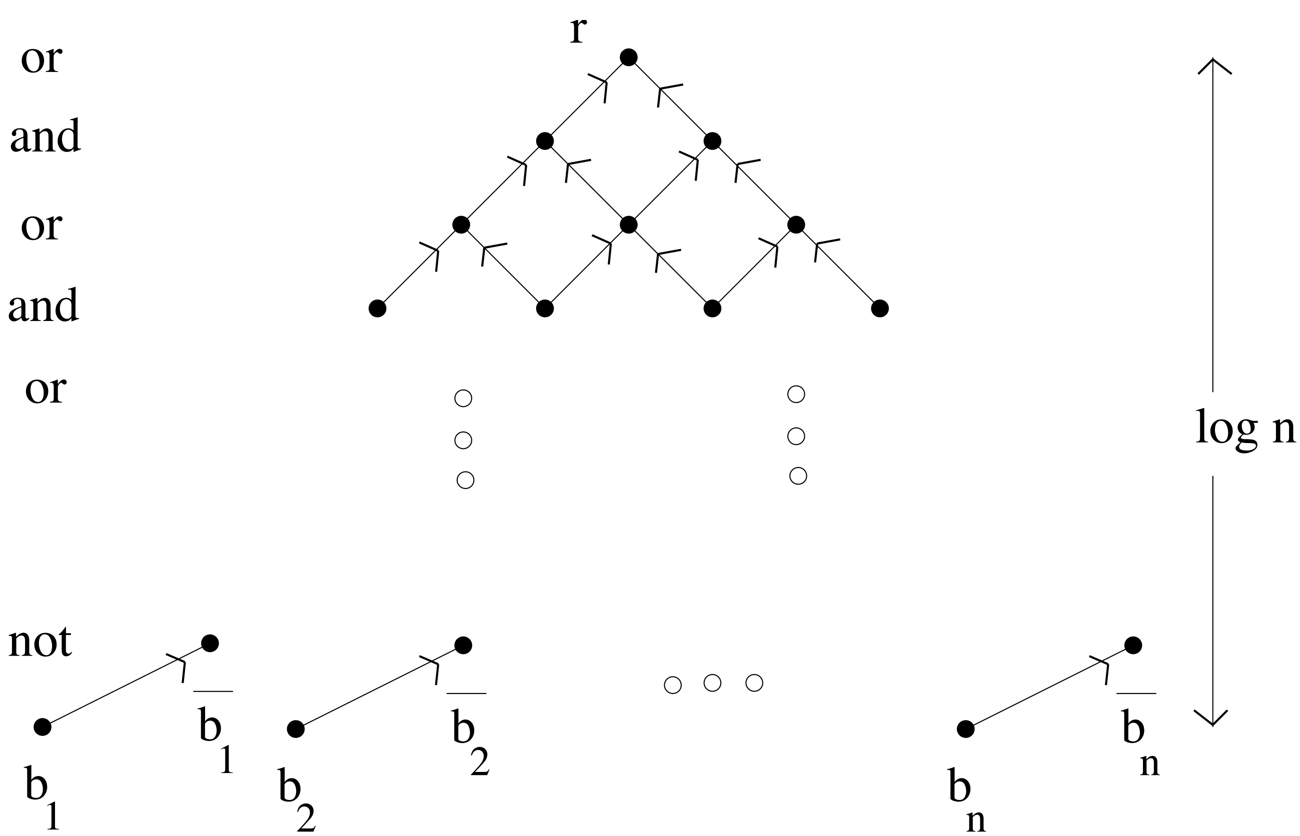
$$\textit{Bounded Width Branching Programs} \quad = \quad \mathrm{NC}^1$$

Proof:

Given an NC^1 circuit, simulate it using the above lemmas.

We multiply the size of the branching programs by 4 as we go up one level.

Total size is $4^{O(\log n)} = n^{O(1)}$



$\square$