14.1 First-Order Resolution

Last time, we discussed Algorithm 14.6: The Complete FO Resolution Algorithm.

Example 14.1

Let $\psi_1 = 1\forall x\exists y(f(x) = y)$. Show $\psi_1 \in \text{FO-VALID}$.

First, let $\phi = \neg \psi_1 = \neg(1\forall x\exists y(f(x) = y)) = \exists x\forall y(f(x) \neq y)$. Then Skolemize $\phi$ (replacing all instances of $x$ by a constant $c$) to obtain $\phi_s = \forall y(f(c) \neq y)$.

Now we’re a little stuck, because it appears we only have one clause, and we need at least two clauses in order to do any resolution. So what do we do?

**Jacob’s suggestion:** add the clause $(z = z)$.

Great! For FO Logic with Equality, we always have this clause – the reflexive law for equality – as an axiom – plus a few others I will mention later.

Thus, ClauseSet $= \{(\neg f(c) = y), (z = z)\}$, or in prefix notation, $\{(\neg = (f(c), y)), (= (z, z))\}$.

We use the Unification Algorithm to compute the mgu of $\{(\neg f(c), y), (= (z, z))\}$. Since the two atoms first differ on $f(c)$ and $z$, we set our substitution $s = [f(c)/z]$. Then the clauses differ again on $f(c)$ and $y$, so we update the substitution to $s = [f(c)/z, f(c)/y]$. This is the most general unifier.

Now we have $c_1 s = (\neg(f(c) = f(c)))$, and $c_2 s = (f(c) = f(c))$, so we can resolve these two clauses to obtain the empty clause. $\square$

Example 14.2 Let $\psi_2 = \forall x(A(x) \land B(x)) \rightarrow \forall x A(x)$. Prove $\psi_2 \in \text{FO-VALID}$.

Let $\phi = \neg \psi_2$, then put $\phi$ into RPF:

$$\phi = \neg \psi_2$$

$$\equiv \neg (\forall x (A(x) \land B(x)) \rightarrow \forall x A(x))$$

$$\equiv \neg (\neg \forall x (A(x) \land B(x)) \lor \forall x A(x))$$

$$\equiv \forall x (A(x) \land B(x)) \land \neg \forall x A(x)$$

$$\equiv \forall x (A(x) \\land B(x)) \land \exists x \neg A(x)$$

$$\equiv \forall x (A(x) \land B(x)) \land \exists y \neg A(y)$$

$$\equiv \exists y \forall x (A(x) \land B(x) \land \neg A(y))$$

Notice that in the last step we chose to move the existential quantifier outside before the universal quantifier in order to make Skolemization simpler. (If we had done it the other way, then we would still be equivalent to $\phi$ thus unsatisfiable.)

Finally, we Skolemize $\phi$ to obtain $\phi_s = \forall x(A(x) \land B(x) \land \neg A(c))$. Then we have three clauses available for resolution, which after renaming of variables are $C_1 = (A(x))$, $C_2 = (B(y))$, and $C_3 = (\neg A(c))$. We will resolve $C_1$ and $C_3$.

We first use the Unification Algorithm to obtain the most general unifier $s = [c/x]$, then have $C_1 s = (A(c))$, and $C_2 s = (\neg A(c))$. We resolve these two clauses to get the empty clause. $\square$
14.2 Compactness Theorem for First Order Logic

The following theorem can be viewed as a corollary of Gödel’s Completeness Theorem (Thm. 11.5).

**Theorem 14.3 (Compactness of FO Logic)** Let $\Gamma \subseteq \mathcal{L}(\Sigma)$ be a collection of first order formulas. If $\Gamma$ is finitely satisfiable (that is, every finite subset of $\Gamma$ is satisfiable), then $\Gamma$ is satisfiable.

**Proof:** Suppose $\Gamma$ is finitely satisfiable, and assume for contradiction that $\Gamma$ is not satisfiable. Every formula is satisfiable iff its universal closure is satisfiable iff the Skolemization of its universal closure is satisfiable, so we can assume without loss of generality that $\Gamma$ consists only of universal sentences. In fact, we can assume that each such formula is in CNF and we consider them a single clause at at time, i.e., all the formulas in $\Gamma$ are of the form $\forall x_1, \ldots, x_k C$ where $C$ is a clause.

We saw in the proof of Herbrand’s Theorem that $\Gamma$ is satisfiable iff $E(\Gamma)$ is satisfiable, where

$$E(\Gamma) = \{ C[t_1/x_1, t_2/x_2, \ldots, t_k/x_k] \mid \forall x_1, \ldots, x_k C \in \Gamma; t_1, \ldots, t_k \text{ are closed terms} \}$$

That is, $E(\Gamma)$ is the collection of (propositional) formulas in which every possible closed-term substitution is made for every variable in every formula of $\Gamma$.

Since we assumed $\Gamma$ was not satisfiable, then $E(\Gamma)$ is not either. But $E(\Gamma)$ is a collection of propositional formulas, so by the Compactness Theorem for Propositional Logic, $E(\Gamma)$ is not finitely satisfiable. This contradicts the fact that $\Gamma$ was finitely satisfiable, as $E(\Gamma)$ has some finite subset of formulas that are unsatisfiable, and this corresponds to a finite subset of formulas in $\Gamma$ that are unsatisfiable. \qed

We can use the Completeness Theorem more directly to obtain an alternative proof:

**Proof:** We show the contrapositive, that if $\Gamma$ is not satisfiable then it is not finitely satisfiable. If $\Gamma$ is not satisfiable, then by the Completeness Theorem, there exists a finite length resolution proof of the empty clause from some finite set of assumptions $\{ \gamma_1, \gamma_2, \ldots, \gamma_n \} \subseteq \Gamma$. This finite subset of $\Gamma$ is not satisfiable, so $\Gamma$ is not finitely satisfiable. \qed

14.3 Applications of First-Order Compactness

Let $\Sigma_{\text{graph}} = (E^2;)$ be the standard graph vocabulary.

**Claim 14.4** There is no first order formula $\varphi$ that expresses the property of graph-connectedness. More formally, there does not exist $\varphi$ such that $\text{Mod}(\varphi) = \{ G \in \text{STRUC}[\Sigma_{\text{graph}}] \mid G \text{ is connected} \}$, where $\text{Mod}(\varphi)$ is the set of all models satisfying $\varphi$.

**Proof:** Suppose $\varphi_c$ is a formula expressing graph-connectedness as described above. Let $\Gamma = \{ \varphi_c, \text{dist}_{\geq 1}(s, t), \text{dist}_{\geq 2}(s, t), \ldots \}$. $\Gamma$ is unsatisfiable, because in order to satisfy it, a graph must have two vertices $s$ and $t$ that have $\text{dist}(s, t) \geq n$ for every natural $n$, while also being connected. But every finite subset of $\Gamma$ is satisfiable; for example, let $\Gamma_{0} = \{ \text{dist}_{\geq n_1}(s, t), \text{dist}_{\geq n_2}(s, t), \ldots, \text{dist}_{\geq n_k}(s, t) \}$. Let $N = \max(n_1, n_2, \ldots, n_k)$, and then any model satisfying $\{ \text{dist}_{\geq N}(s, t), \varphi_c \}$ satisfies $\Gamma_{0}$.

By compactness, since every finite subset of $\Gamma$ is satisfiable, $\Gamma$ should be satisfiable, so the fact that it is not contradicts our assumption that such a $\varphi_c$ exists. \qed

As another application, we construct a “non-standard model of arithmetic.” We denote by $\mathbb{N}$ the standard model of arithmetic, with universe $\{0, 1, 2, \ldots\}$, and $\Sigma_N = \{ \leq 0, s, +, * \}$ where $s$ is the successor function, and all functions and predicates are defined in the expected way. We denote the set of formulas satisfied by this model $Th(\mathbb{N})$ (read “the theory of $\mathbb{N}$”), $Th(\mathbb{N}) = \{ \varphi \in \mathcal{L}(\Sigma_N) \mid \mathbb{N} \models \varphi \}$. 

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Now define $\Gamma = \text{Th}(\mathbb{N}) \cup \{0 \leq c, s(0) \leq c, s(s(0)) \leq c, \ldots\}$. $\Gamma$ is finitely satisfiable, because for any finite subset of formulas we can choose $\mathbb{N}$ as our model, which will contain some $c$ large enough to satisfy all of the finitely many added formulas. Then by compactness, $\Gamma$ is satisfiable, so has some model $\mathbb{N}^*$, which we call a “non-standard model of $\text{Th}(\mathbb{N})$.”

We can think of this model as $\mathbb{N}$, along with a new element $c$ larger than any elements of $\mathbb{N}$. We must also include elements like $c - 1, c + 1, 2c, c^2, \lfloor c/2 \rfloor, \lfloor \sqrt{c} \rfloor$, etc. As $\text{Th}(\mathbb{N}) \subseteq \Gamma$, and $\mathbb{N}^* \models \Gamma$, it follows that $\mathbb{N}^* \models \text{Th}(\mathbb{N})$, so all “normal” facts about the natural numbers are also satisfied by this model.

This also tells us that any attempt to define a specific model expressing what we think of as the natural numbers is not possible using only first-order axioms, as any such set of axioms is also be satisfied by some non-standard model.